

Bending Hyperbolic Kaleidoscopes

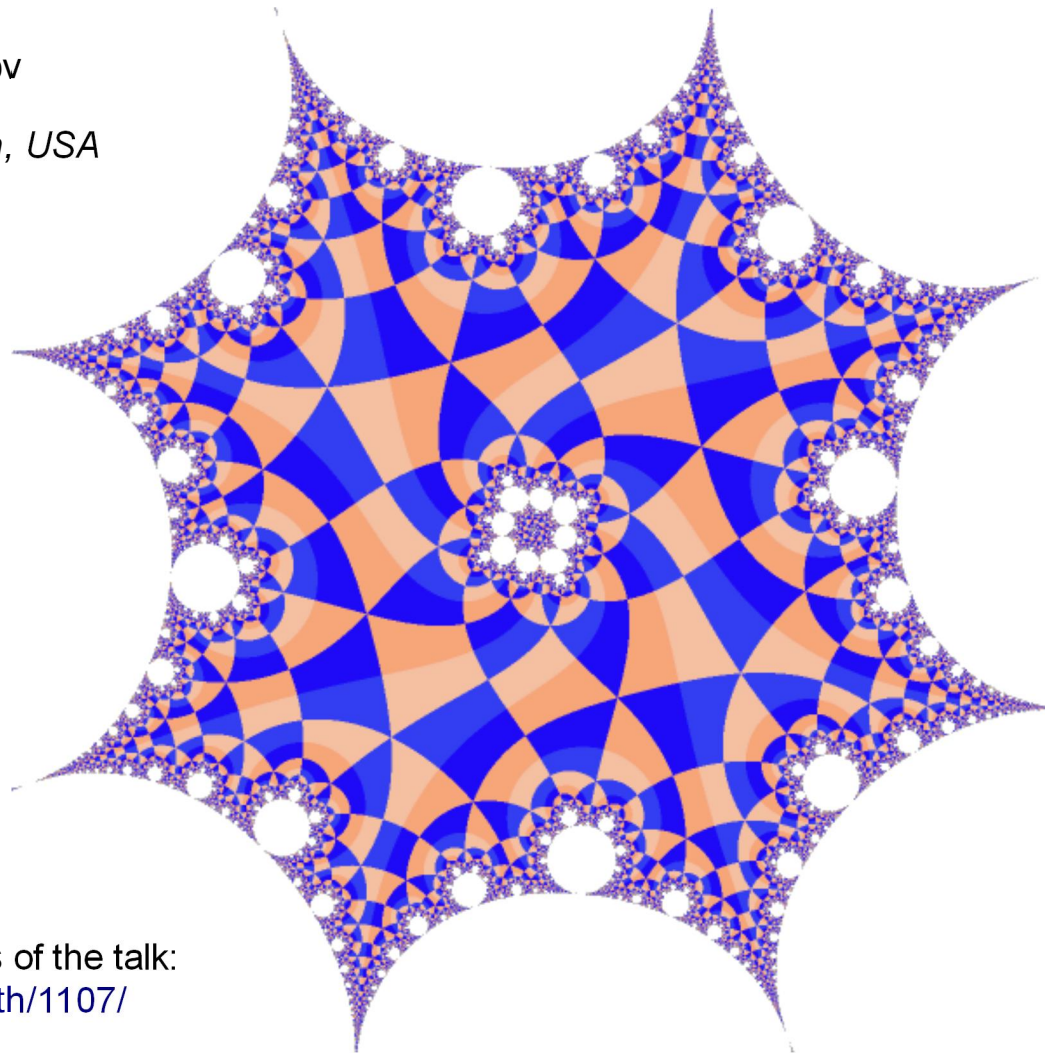
Vladimir Bulatov

Corvallis, Oregon, USA

Bridges Conference

July 27, 2011

Coimbra, Portugal

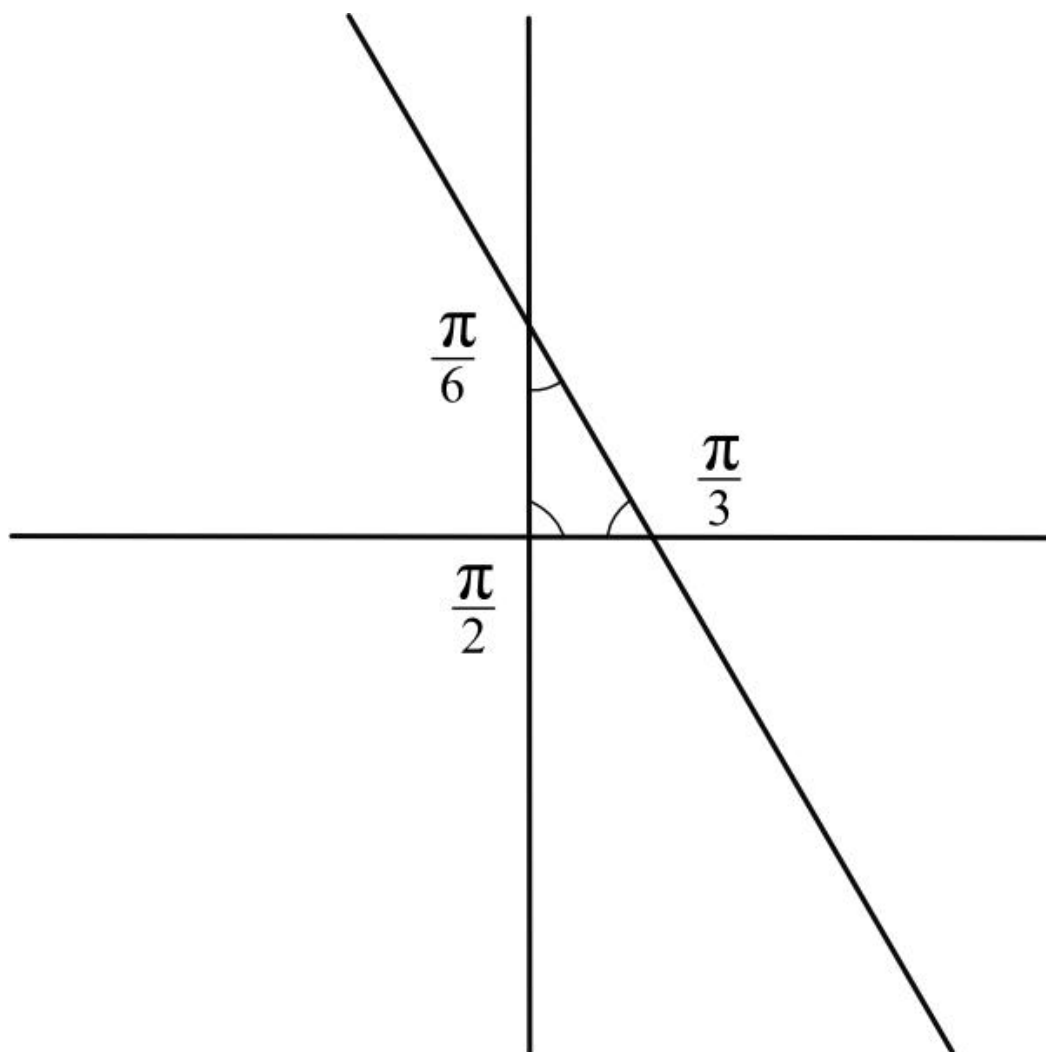


Online address of the talk:
bulatov.org/math/1107/

Outline

- Euclidean Plane kaleidoscopes
- Triangular Hyperbolic kaleidoscopes
- Polygonal hyperbolic kaleidoscopes
- Stretching polygonal kaleidoscopes
- Bending polygonal kaleidoscopes in 3D
- Cosets coloring of kaleidoscopes
- Conformal transformations
- Horosphere cross section

Euclidean plane kaleidoscope



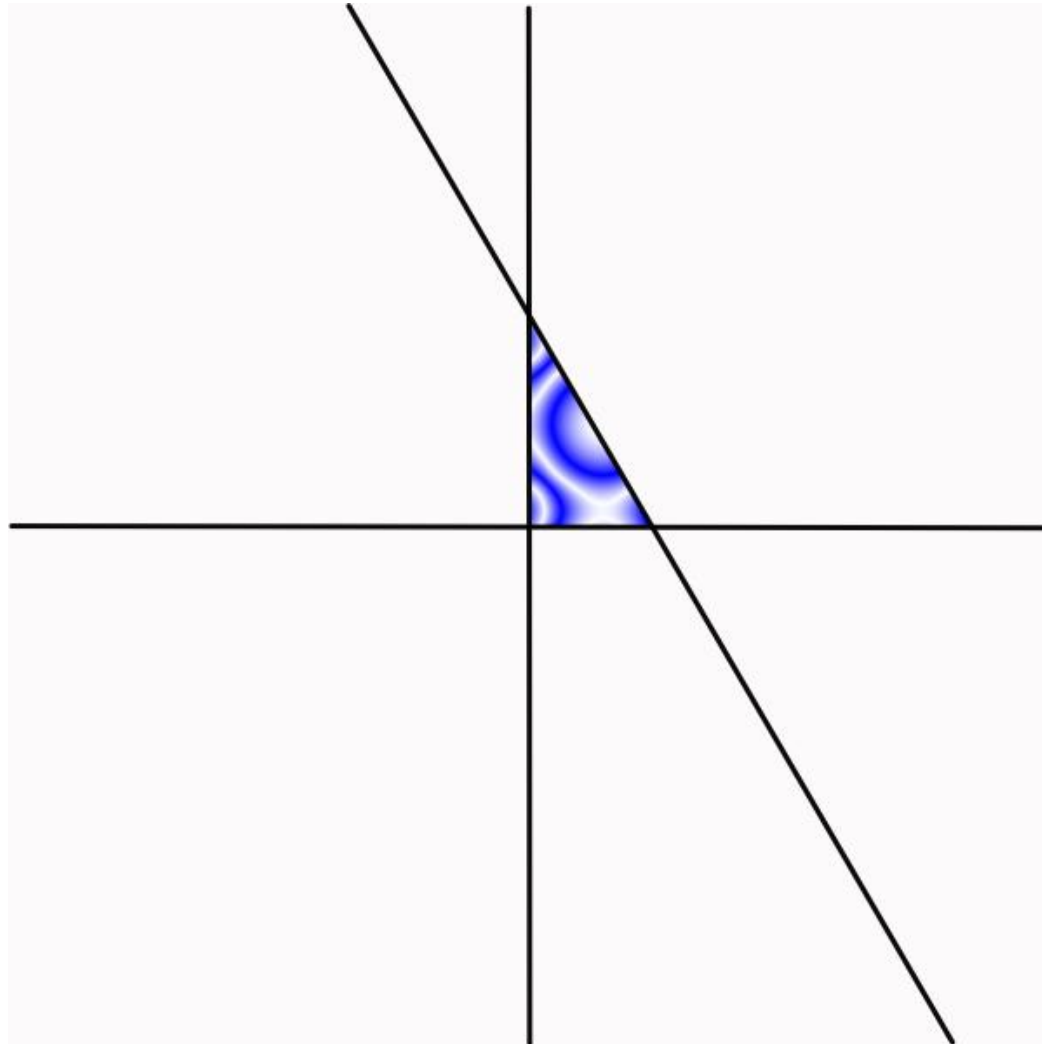
Let's put 3 reflection lines in the plane.

Angles between the lines are fractions of π :

$$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$$

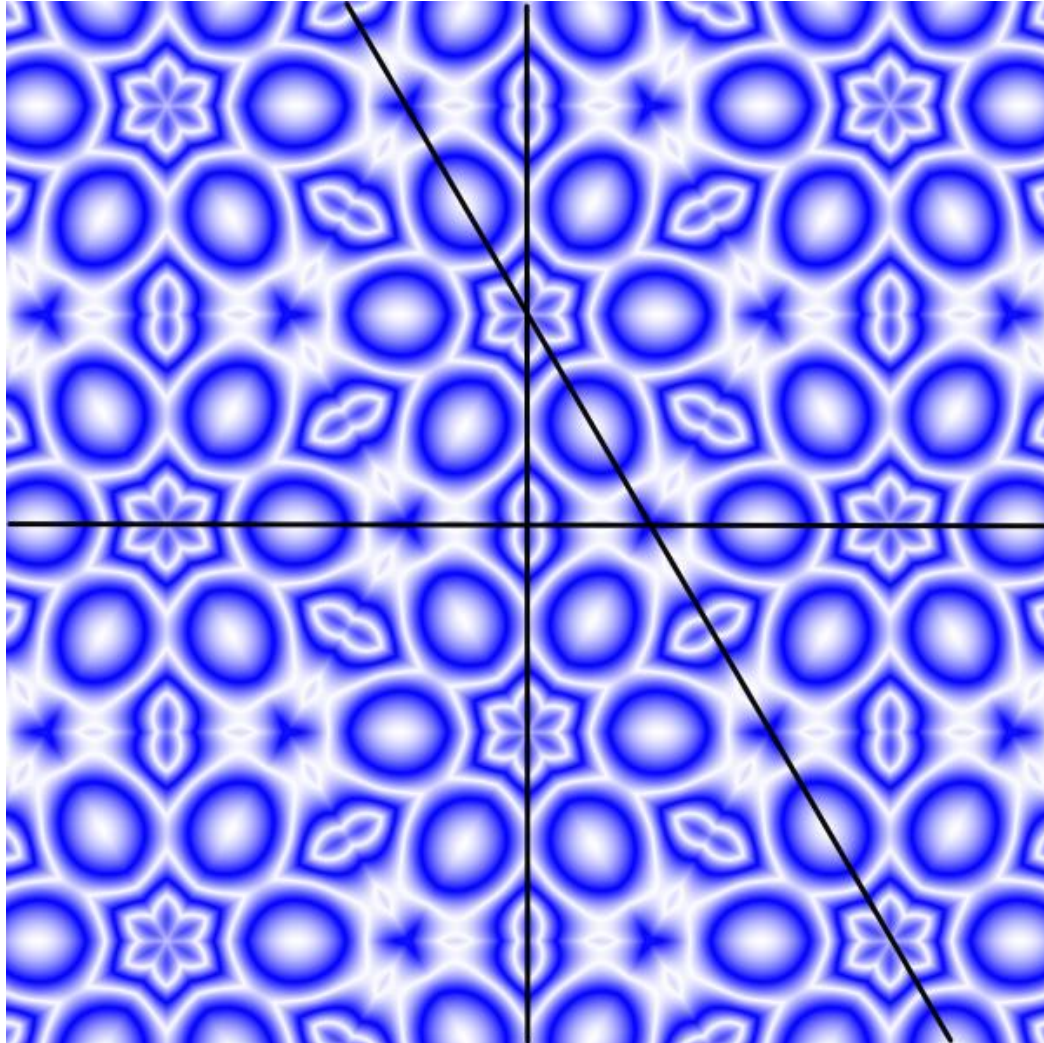
Triangle inside of intersection is
fundamental domain of kaleidoscope.

Euclidean plane kaleidoscope



Let's place arbitrary pattern inside of fundamental domain.

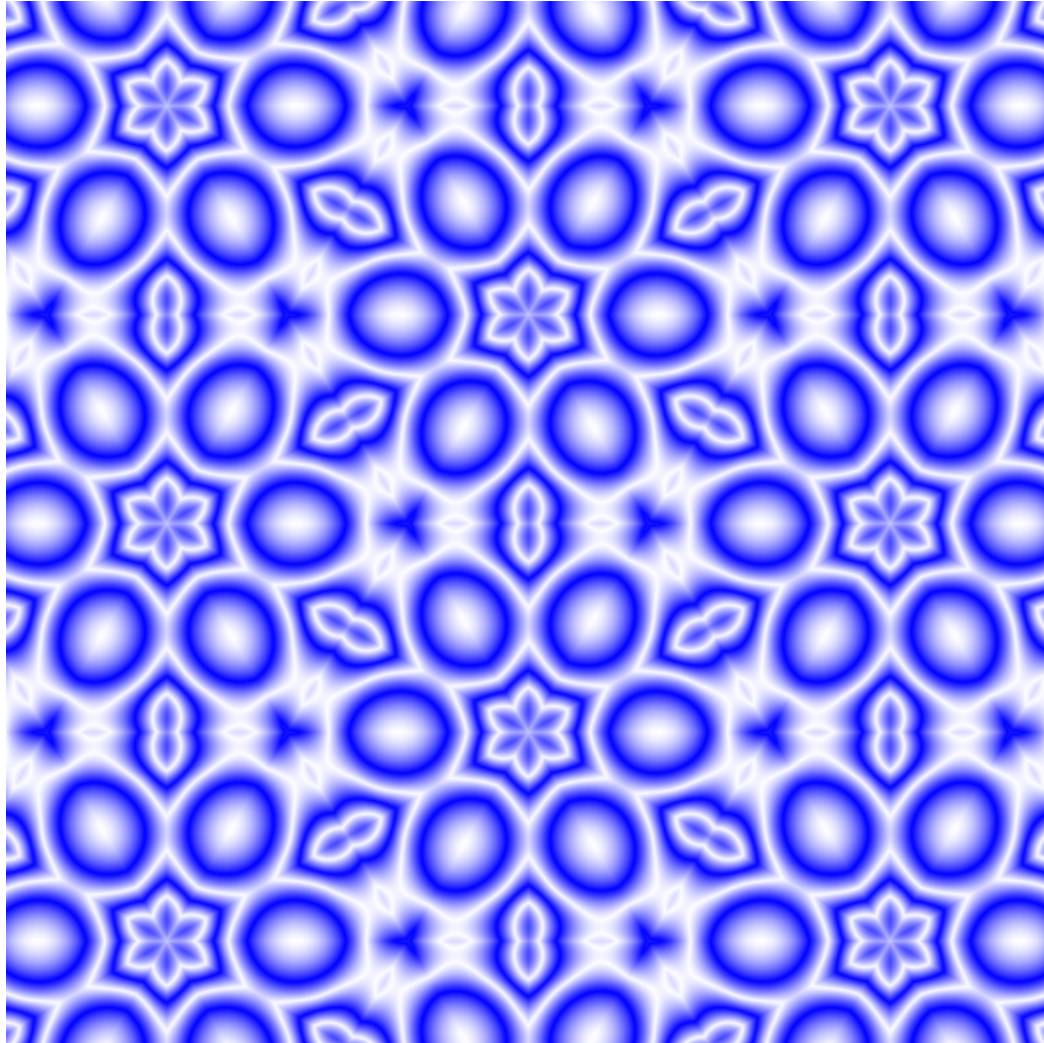
Euclidean plane kaleidoscope



And apply all possible reflections in the mirrors.

Symmetry of this pattern is $*236$.

Euclidean plane kaleidoscope



There are only 4 different kind of euclidean kaleidoscopes.

*236

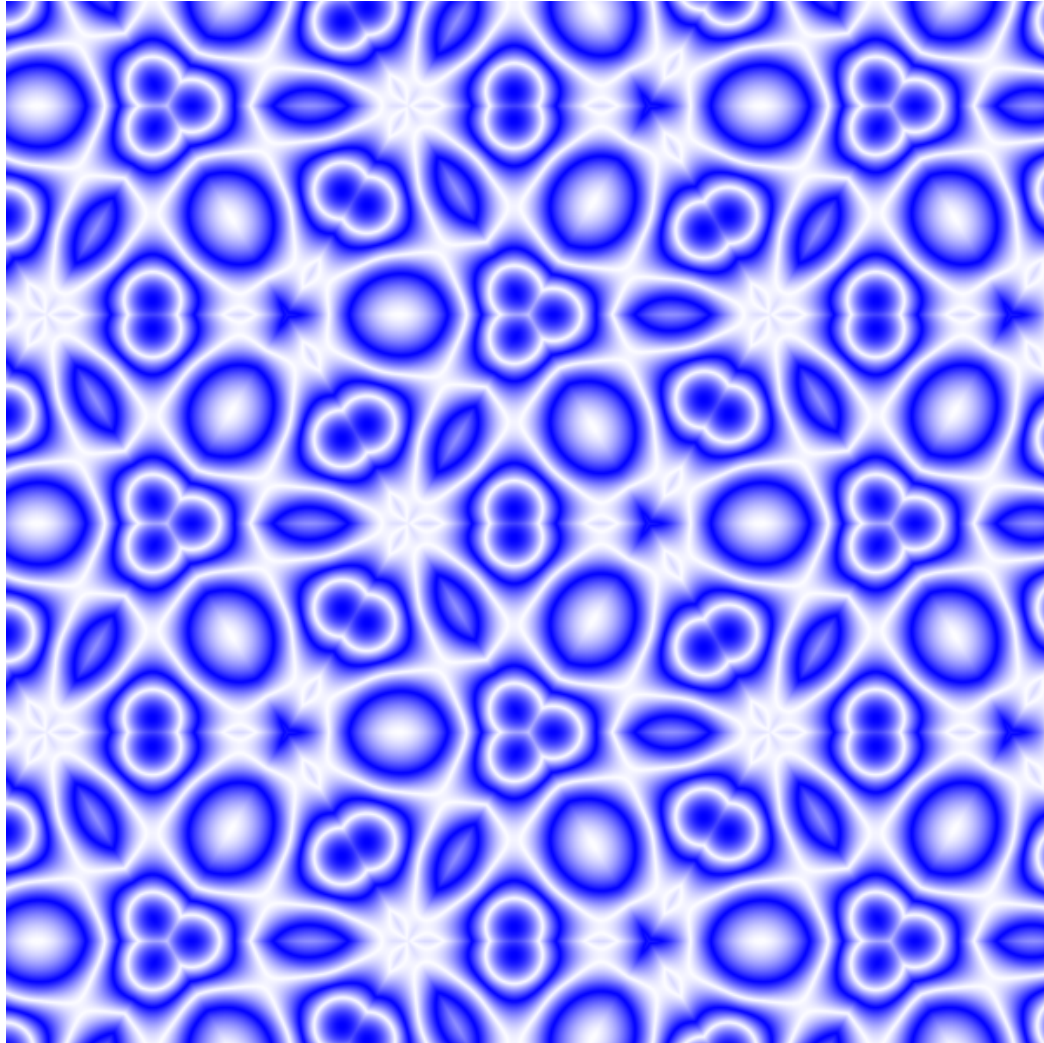
Euclidean plane kaleidoscope



There are only 4 different kind of euclidean kaleidoscopes.

*244

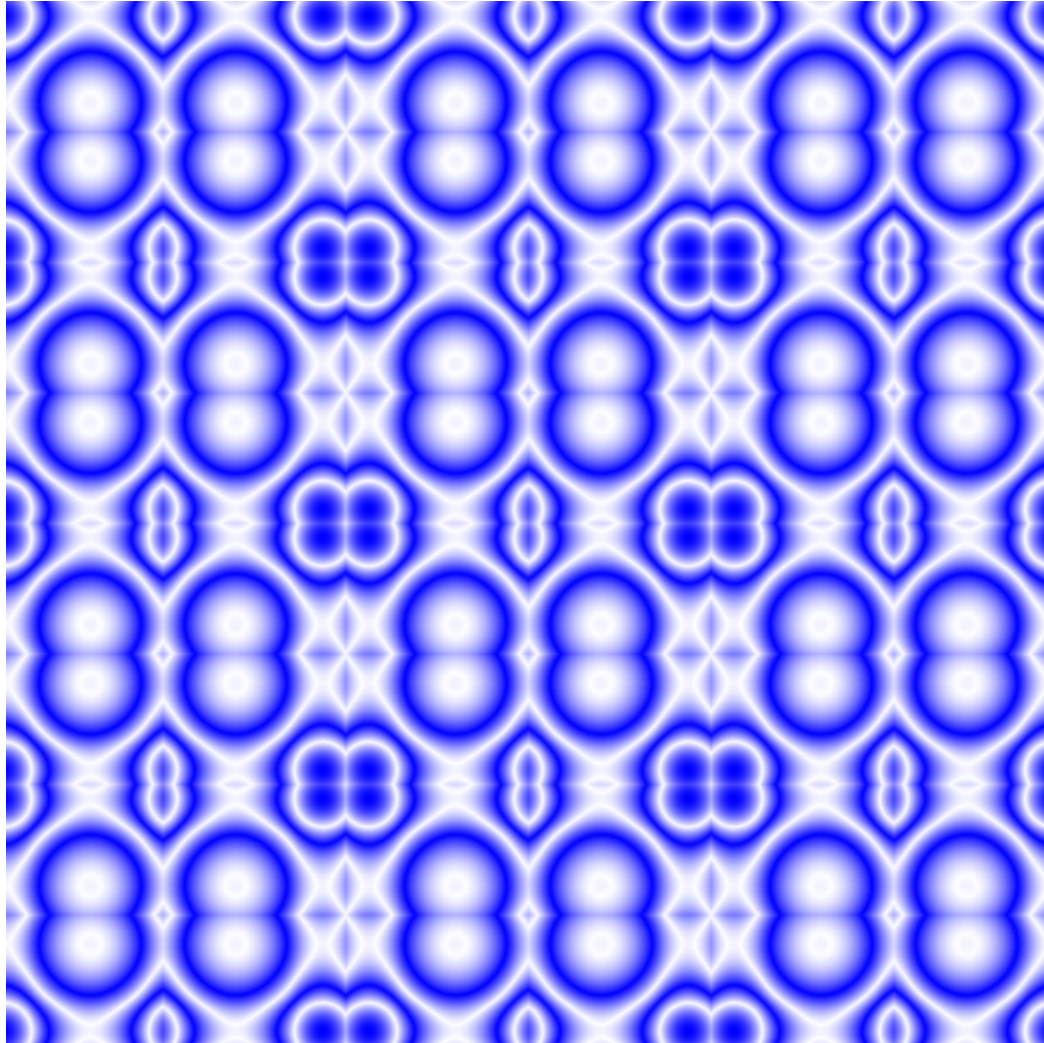
Euclidean plane kaleidoscope



There are only 4 different kind of euclidean kaleidoscopes.

*333

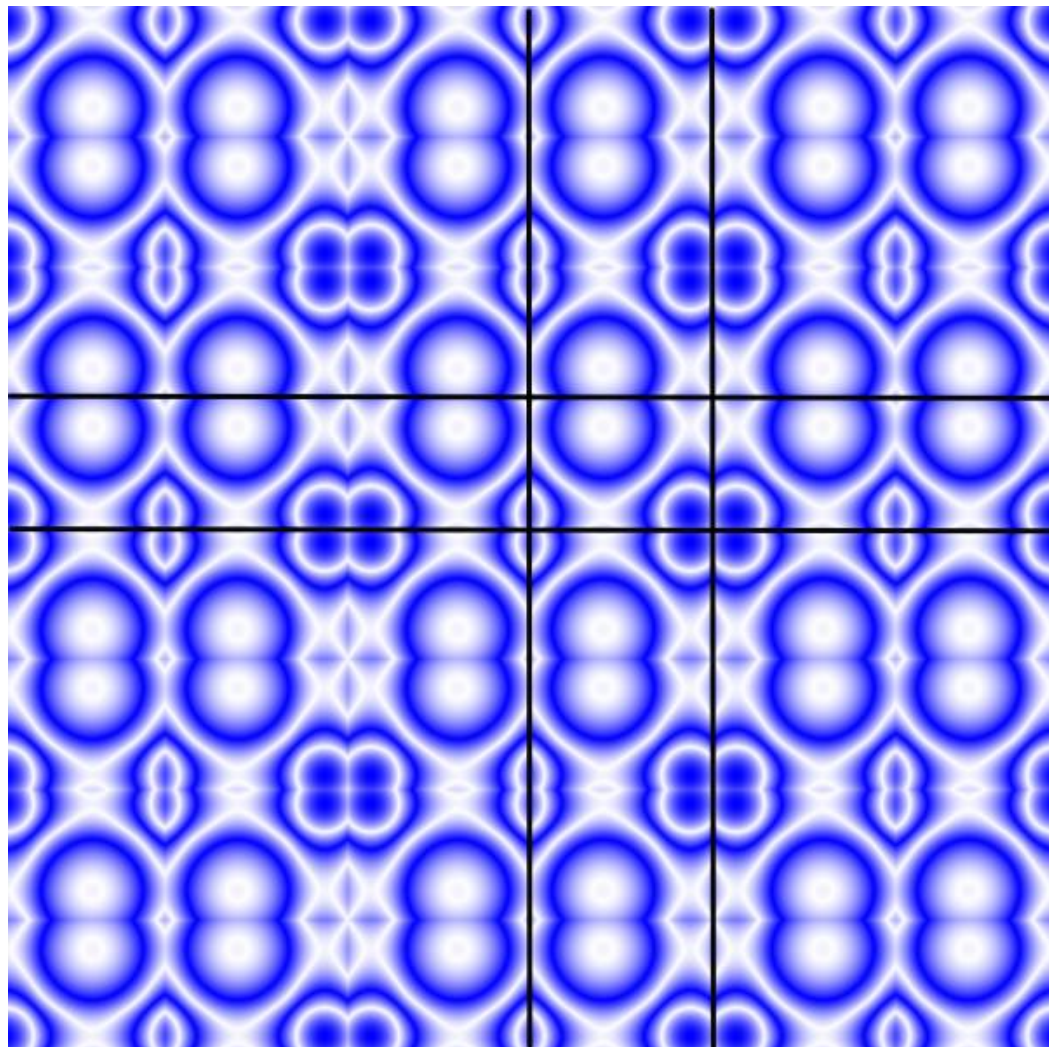
Euclidean plane kaleidoscope



There are only 4 different kind of euclidean kaleidoscopes.

*2222

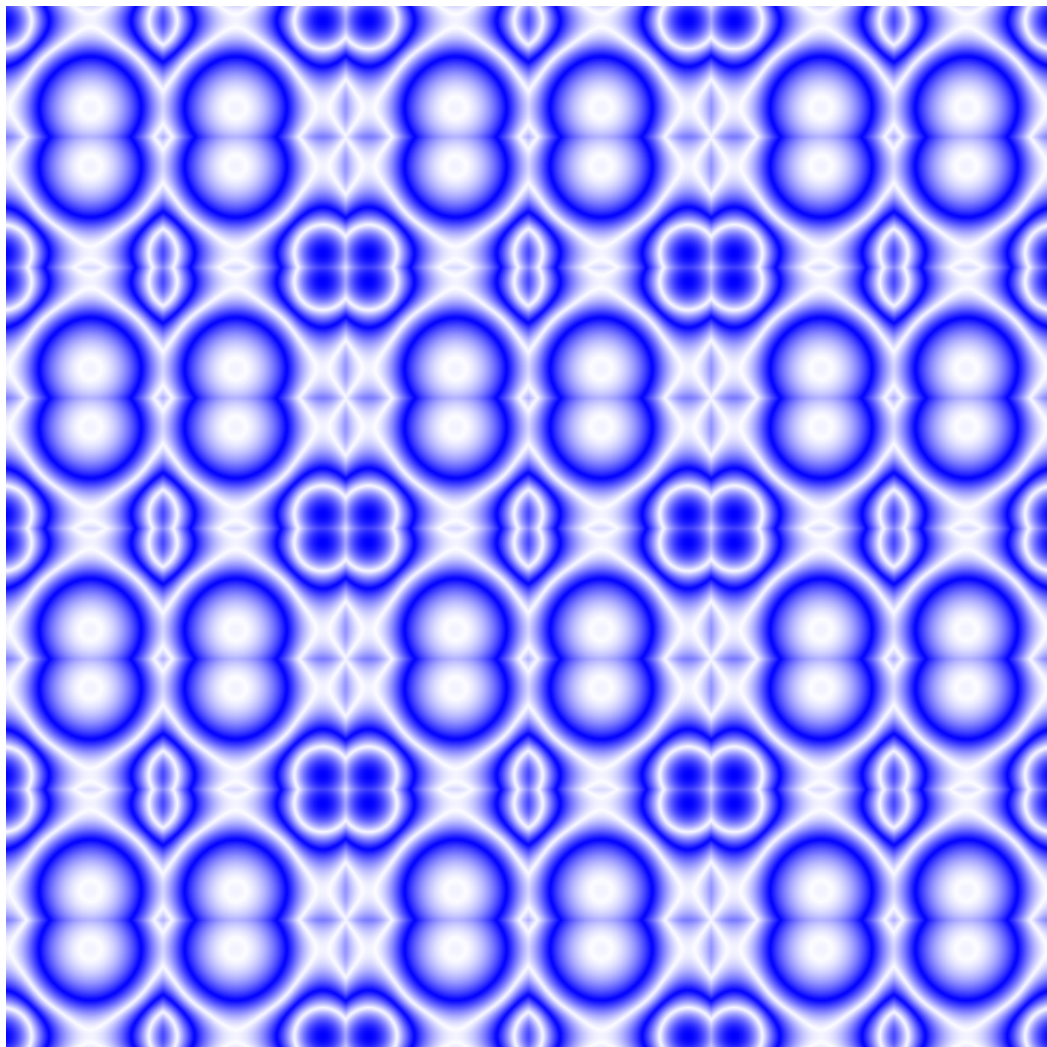
Euclidean plane kaleidoscope



*2222 kaleidoscope is formed by 4 reflection lines. Fundamental domain is rectangle.

It has extra degree of freedom - aspect ratio of rectangle.

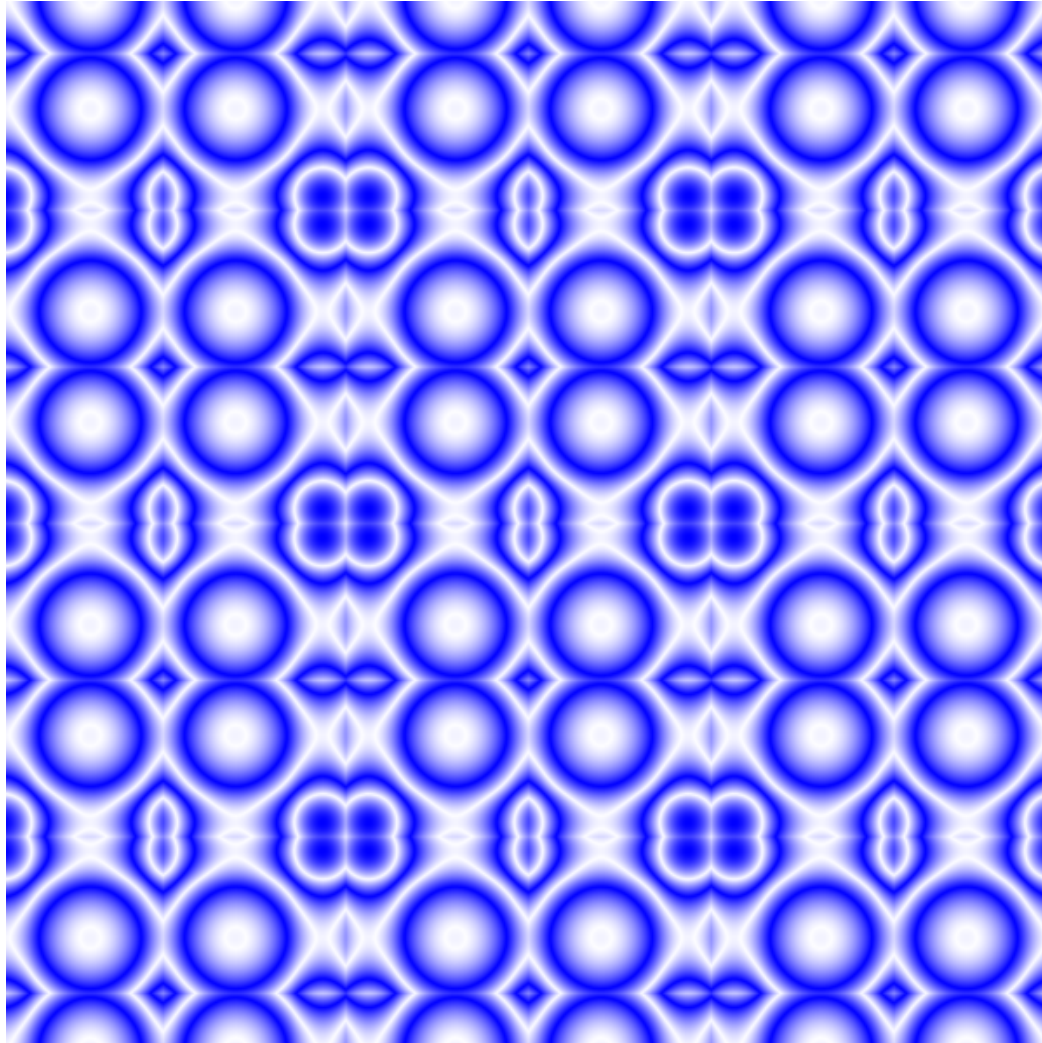
Euclidean plane kaleidoscope



*2222 kaleidoscope is formed by 4 reflection lines. Fundamental domain is rectangle.

It has extra degree of freedom - aspect ratio of rectangle.

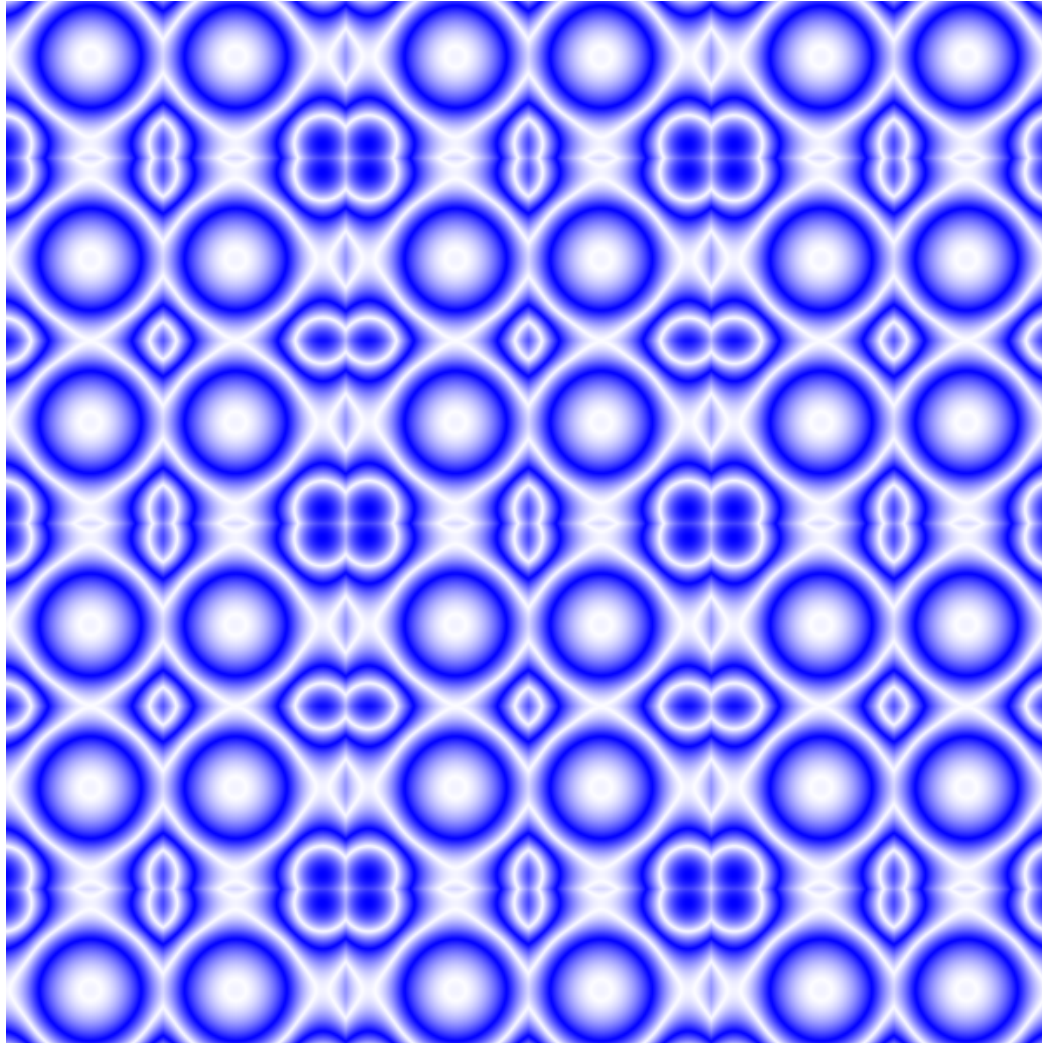
Euclidean plane kaleidoscope



*2222 kaleidoscope is formed by 4 reflection lines. Fundamental domain is rectangle.

It has extra degree of freedom - aspect ratio of rectangle.

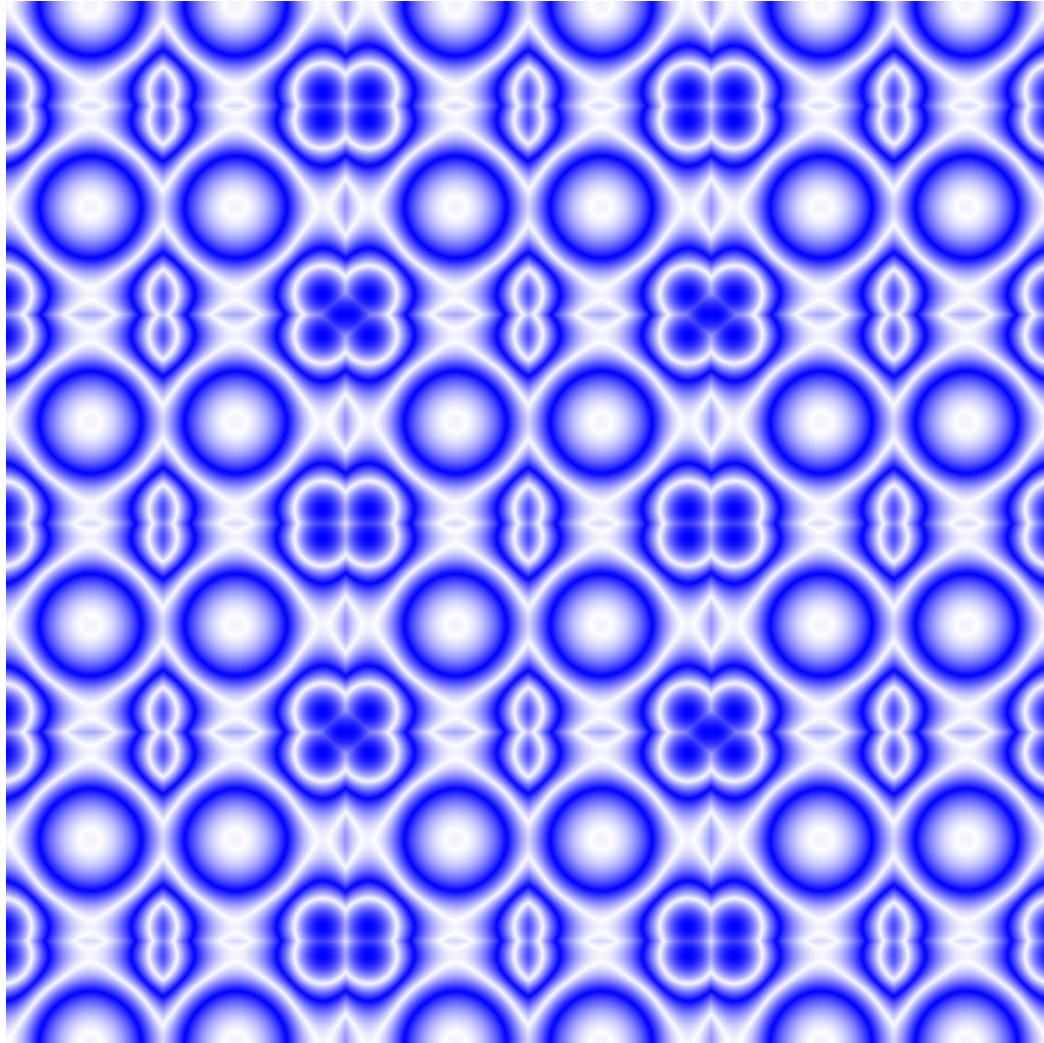
Euclidean plane kaleidoscope



*2222 kaleidoscope is formed by 4 reflection lines. Fundamental domain is rectangle.

It has extra degree of freedom - aspect ratio of rectangle.

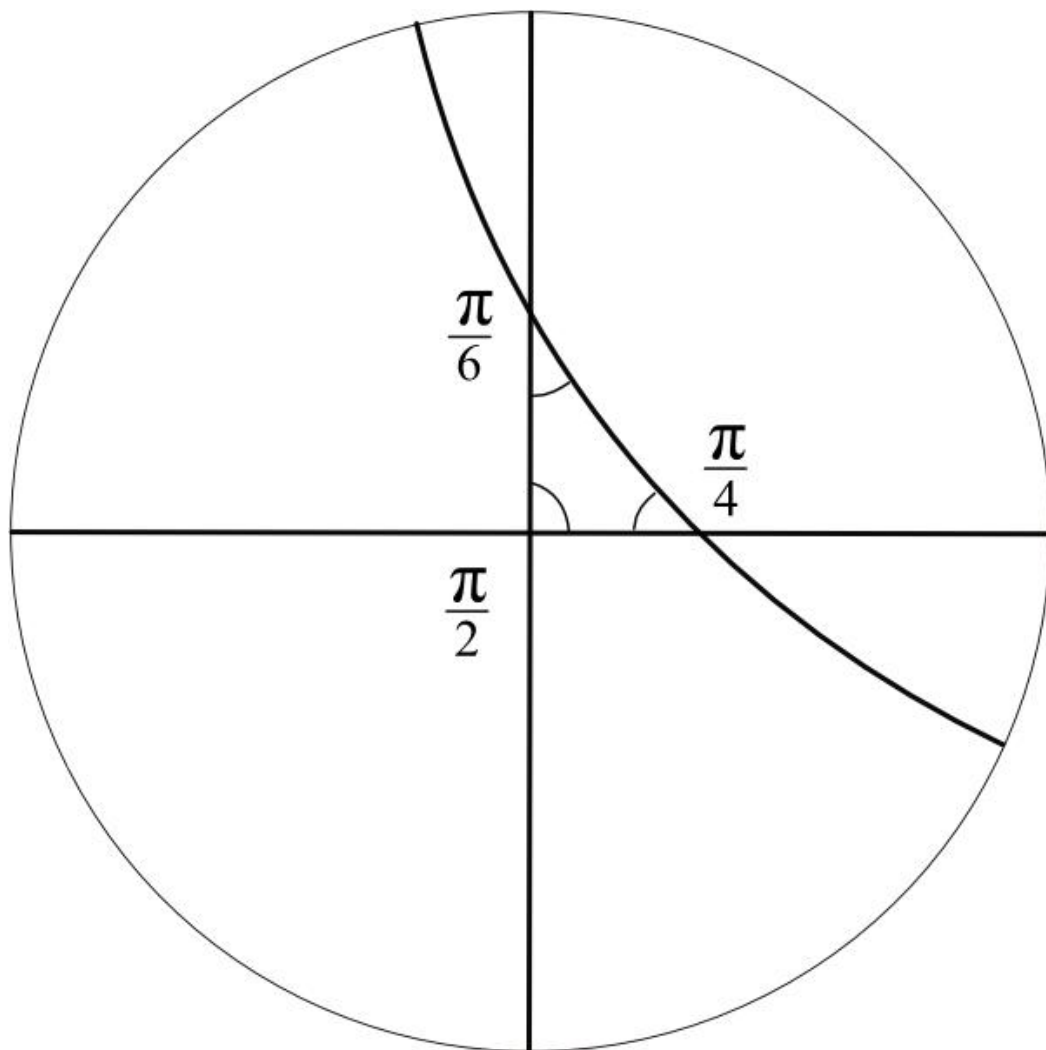
Euclidean plane kaleidoscope



*2222 kaleidoscope is formed by 4 reflection lines. Fundamental domain is rectangle.

It has extra degree of freedom - aspect ratio of rectangle.

Hyperbolic triangle kaleidoscope



Poincare disc model of hyperbolic plane.

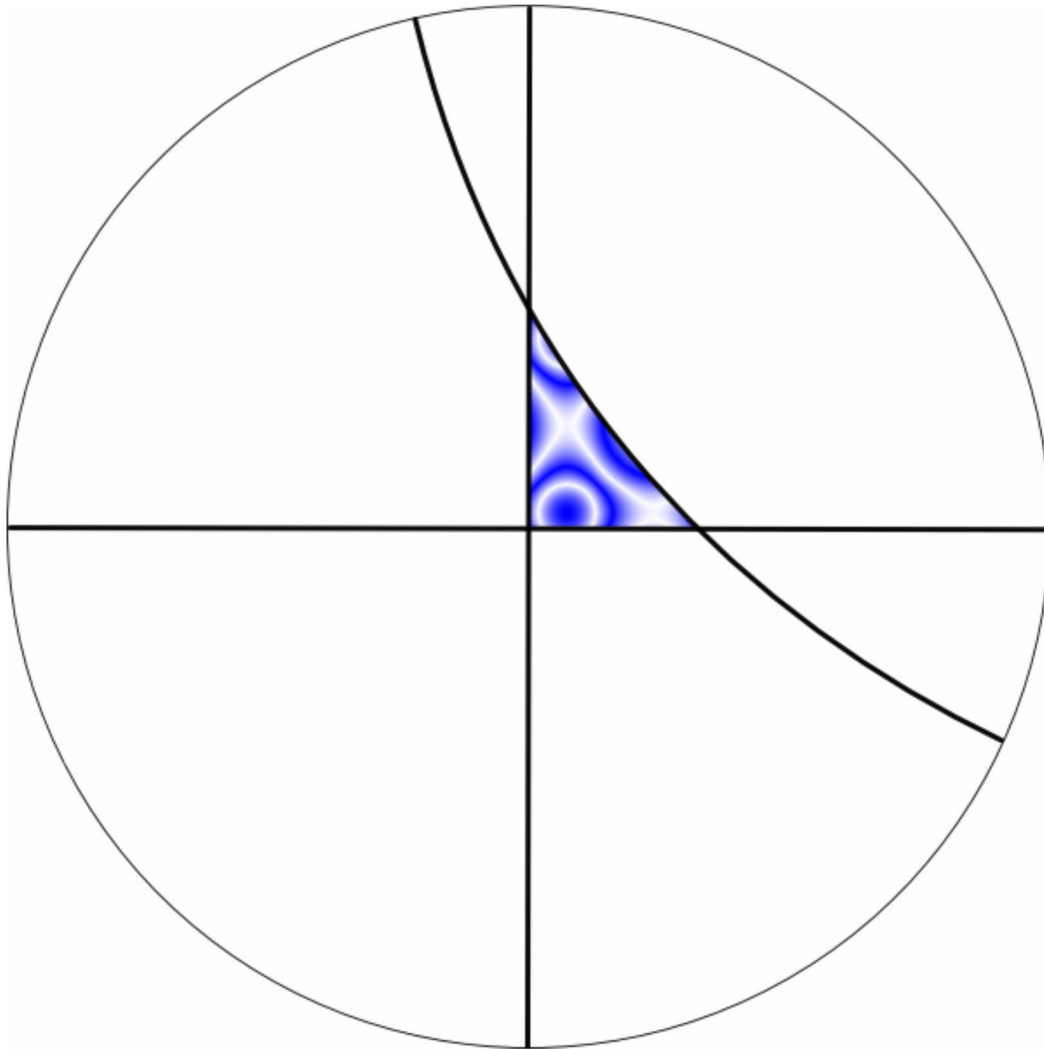
Interior of the unit disk represents the whole hyperbolic plane.

Hyperbolic straight lines are circles (or lines) orthogonal to unit circle.

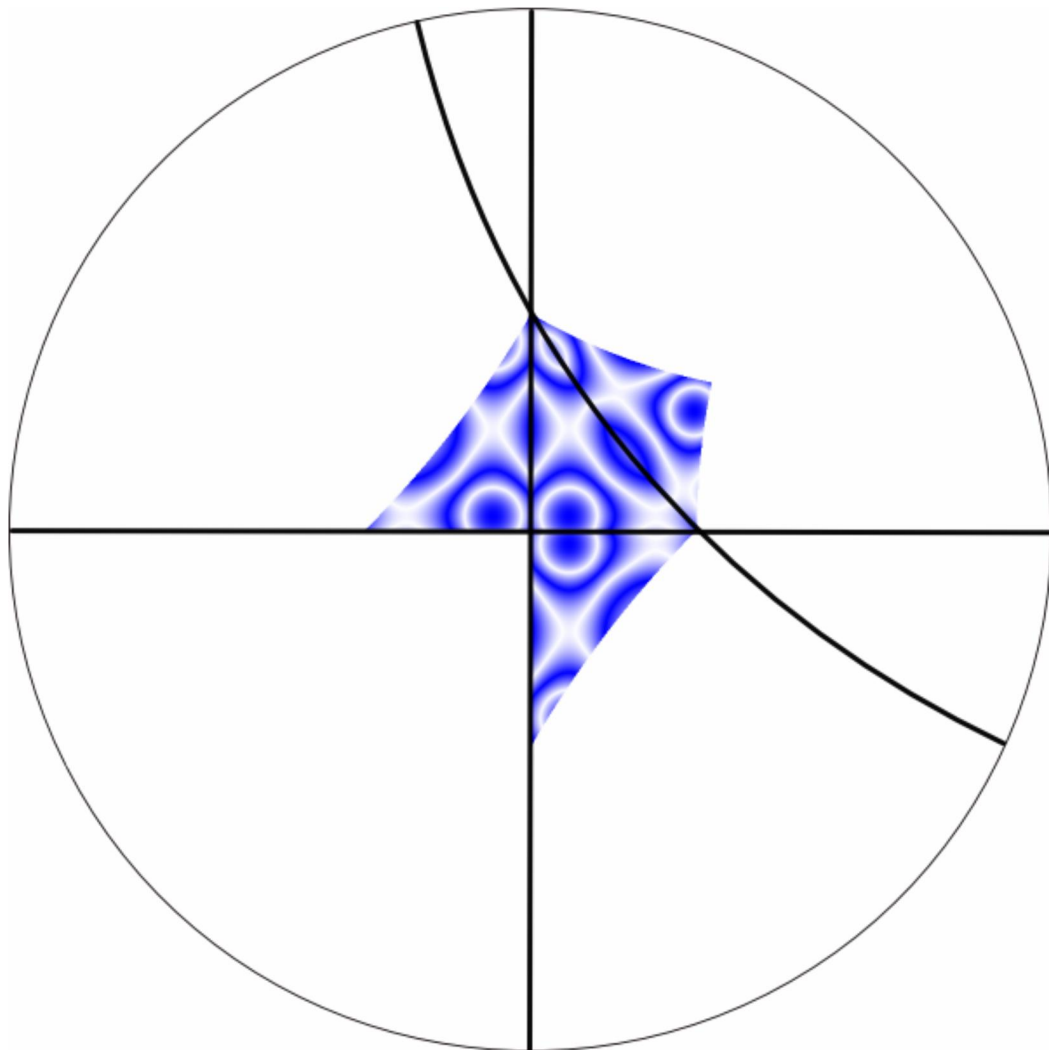
3 reflection lines in the hyperbolic plane
make dihedral angles $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$

Hyperbolic triangle kaleidoscope

Place a pattern into fundamental domain



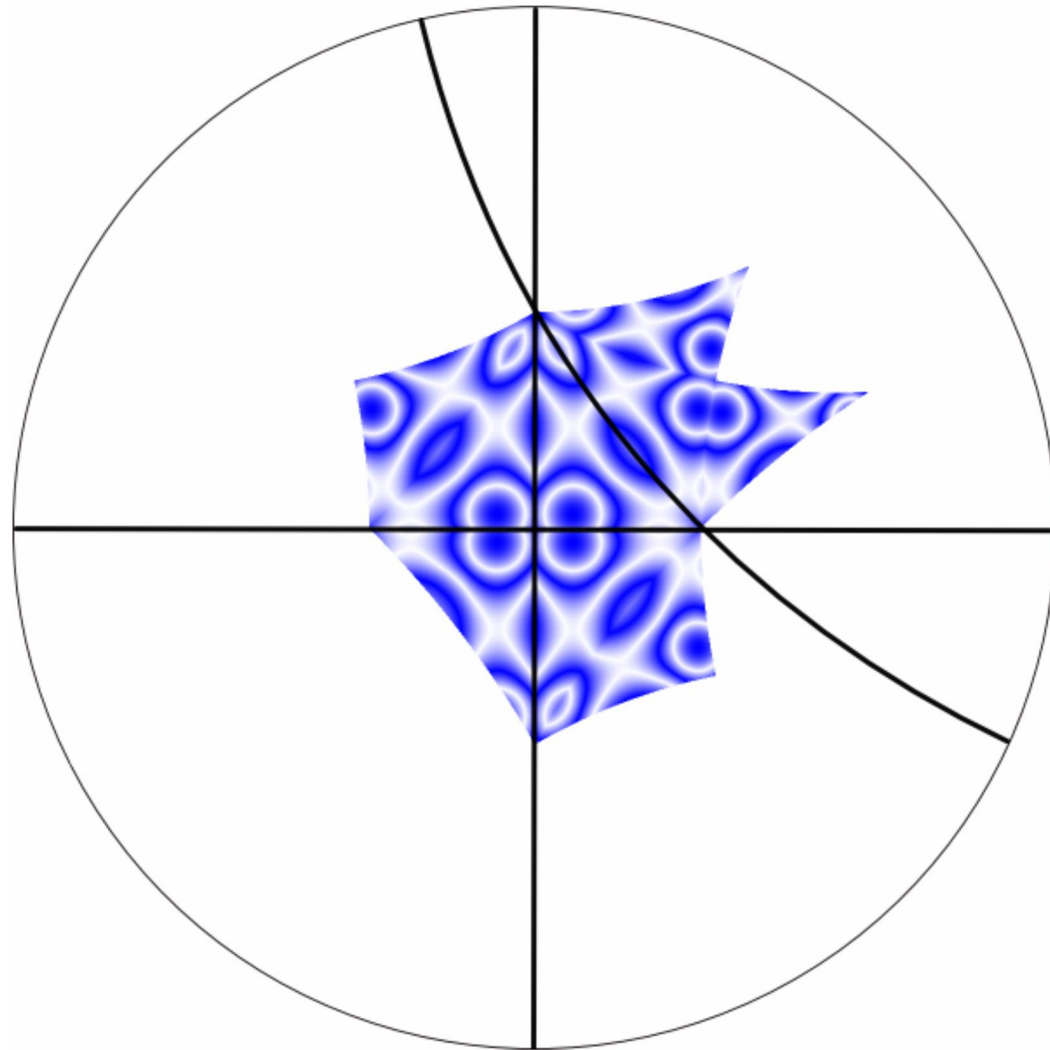
Hyperbolic triangle kaleidoscope



Make one reflection.

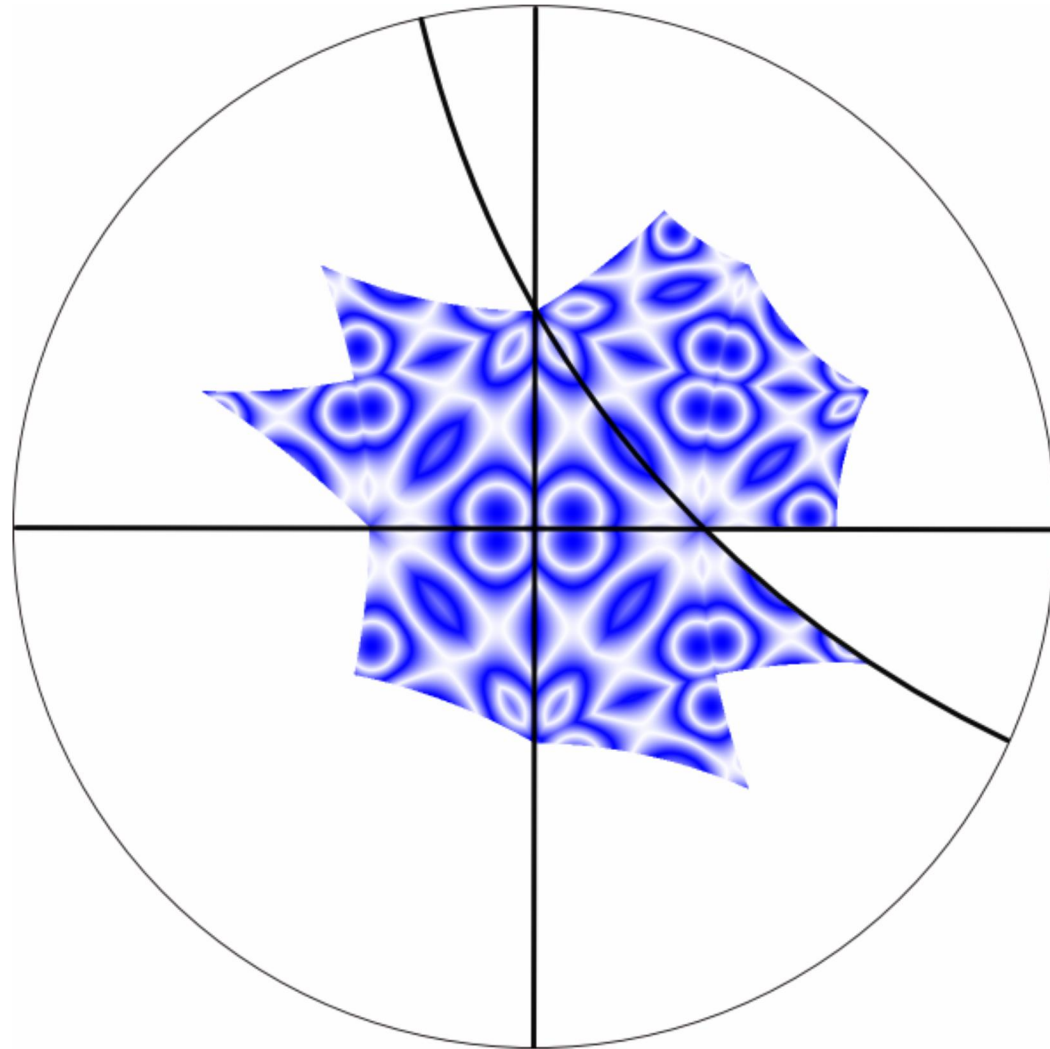
Reflections in lines are replaced with
inversions in circles.

Hyperbolic triangle kaleidoscope



Make 2 reflections

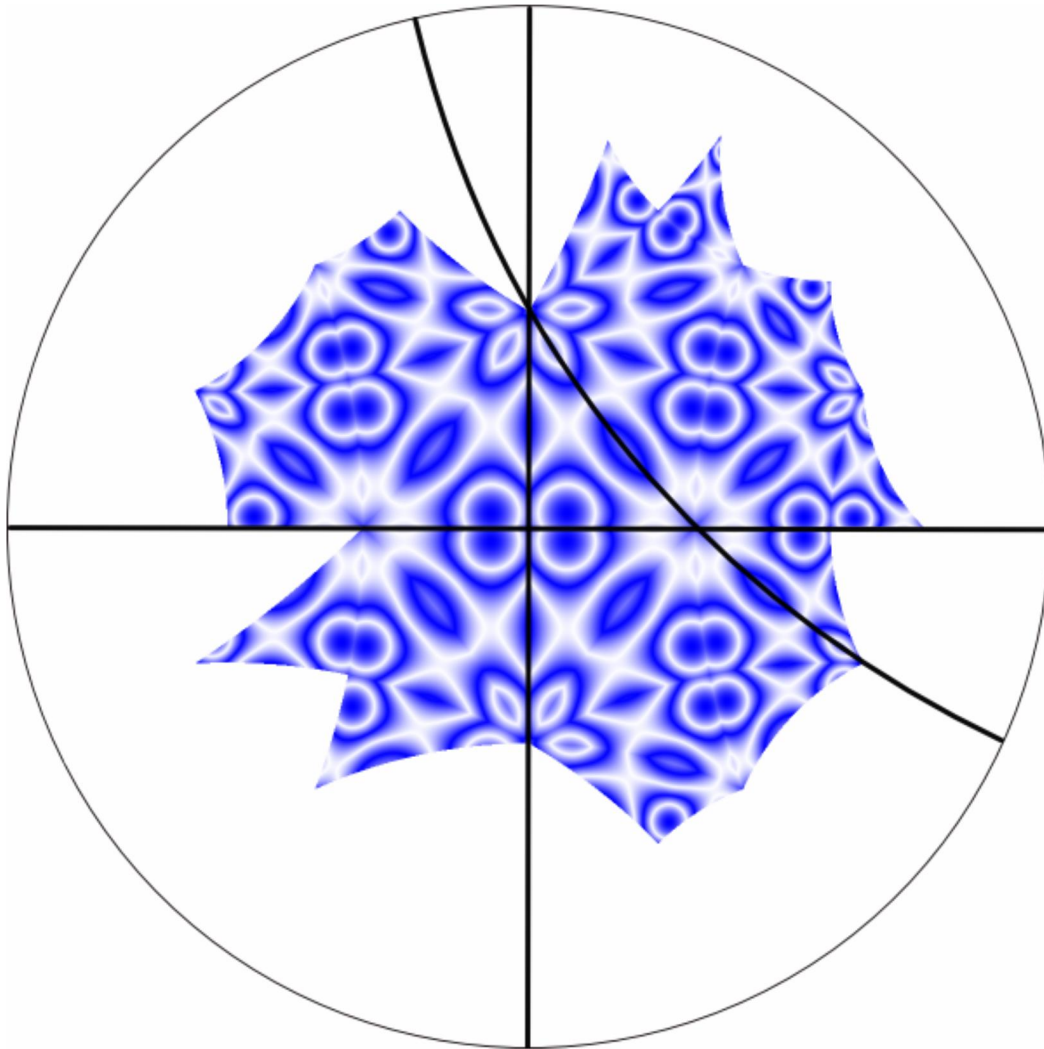
Hyperbolic triangle kaleidoscope



Make 3 reflections

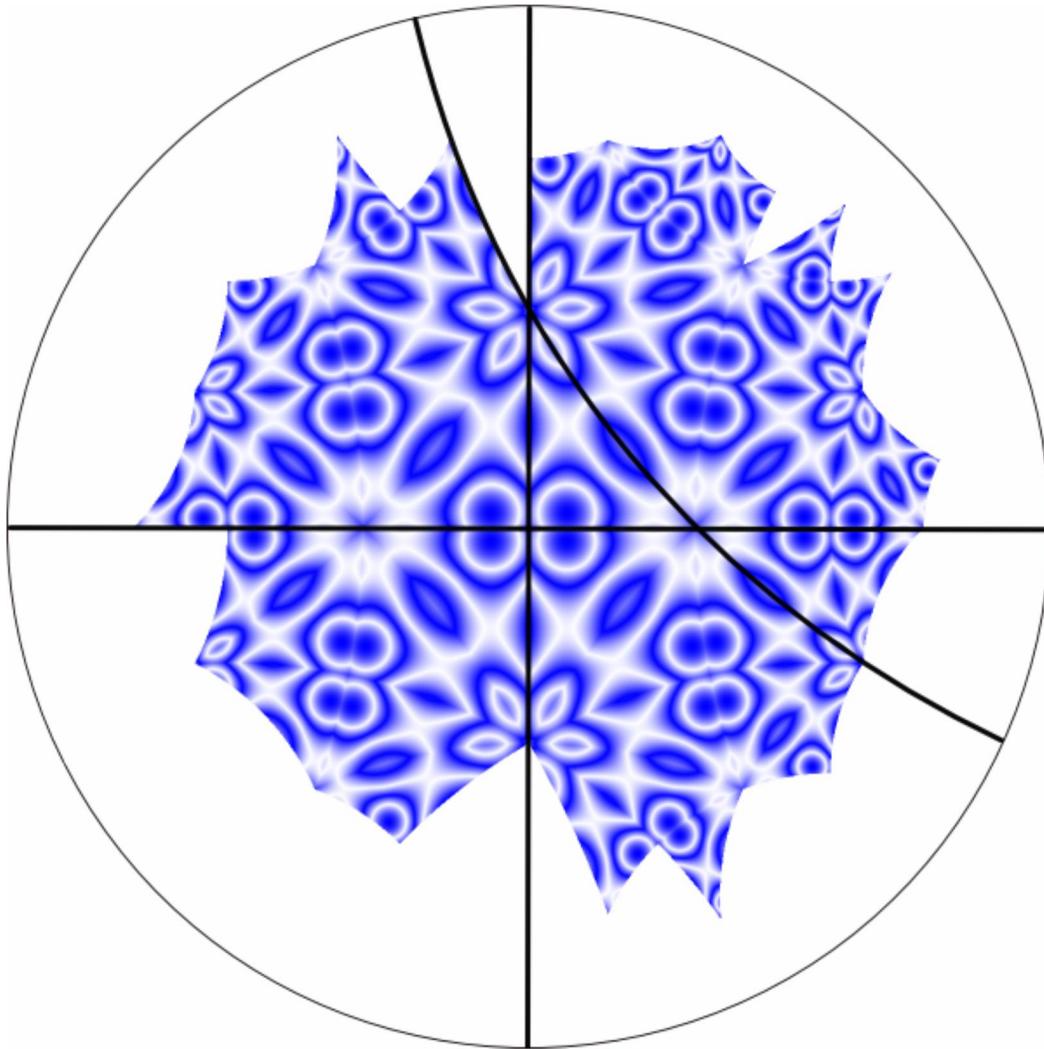
Hyperbolic triangle kaleidoscope

Make 4 reflections



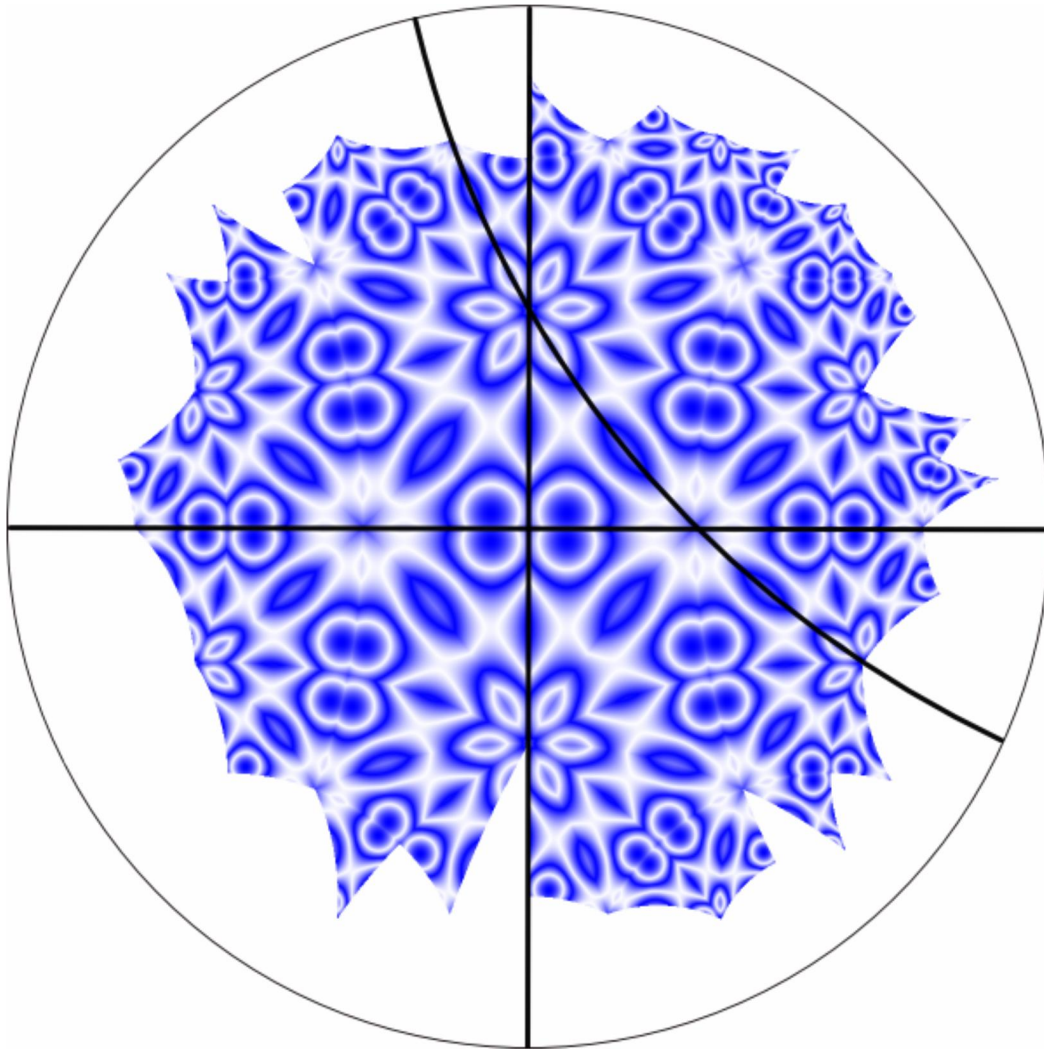
Hyperbolic triangle kaleidoscope

Make 5 reflections



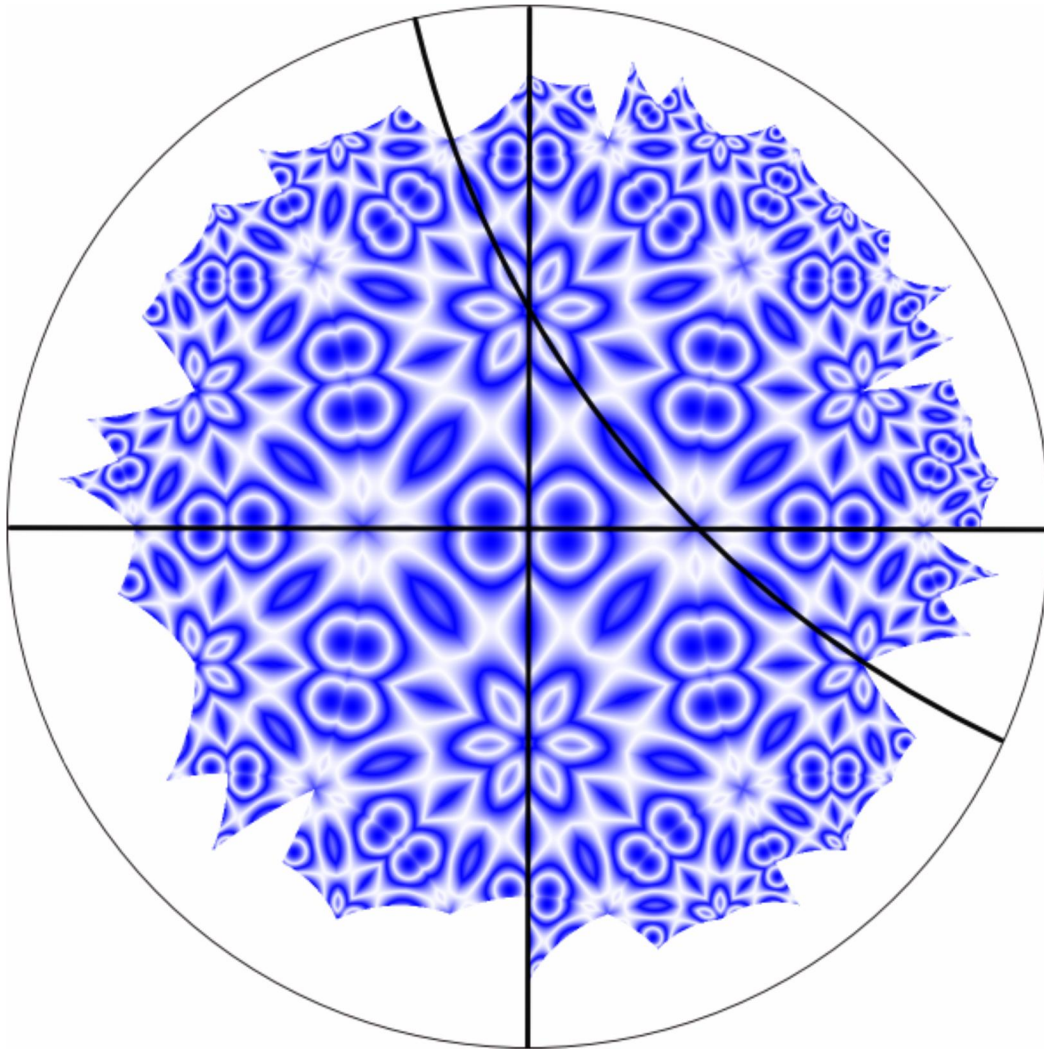
Hyperbolic triangle kaleidoscope

Make 6 reflections



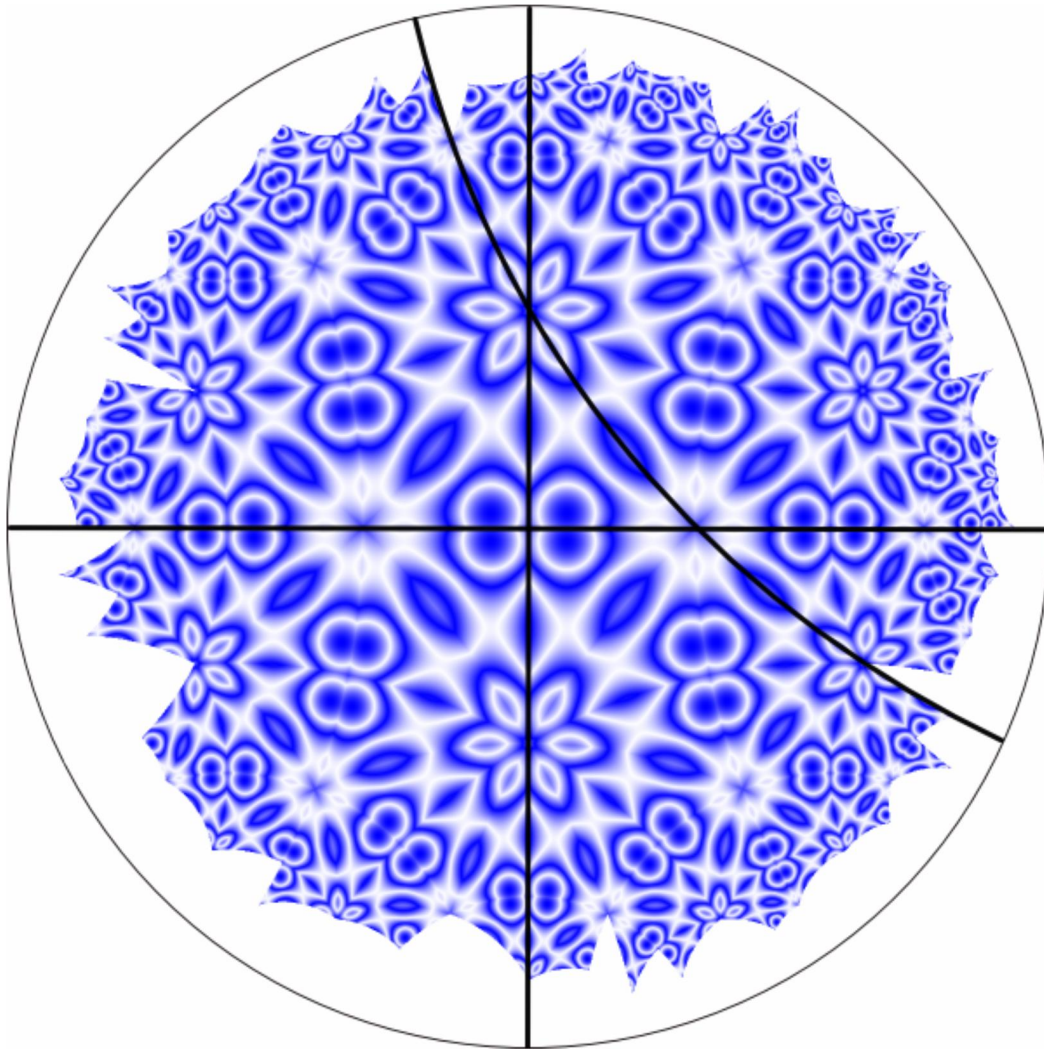
Hyperbolic triangle kaleidoscope

Make 7 reflections



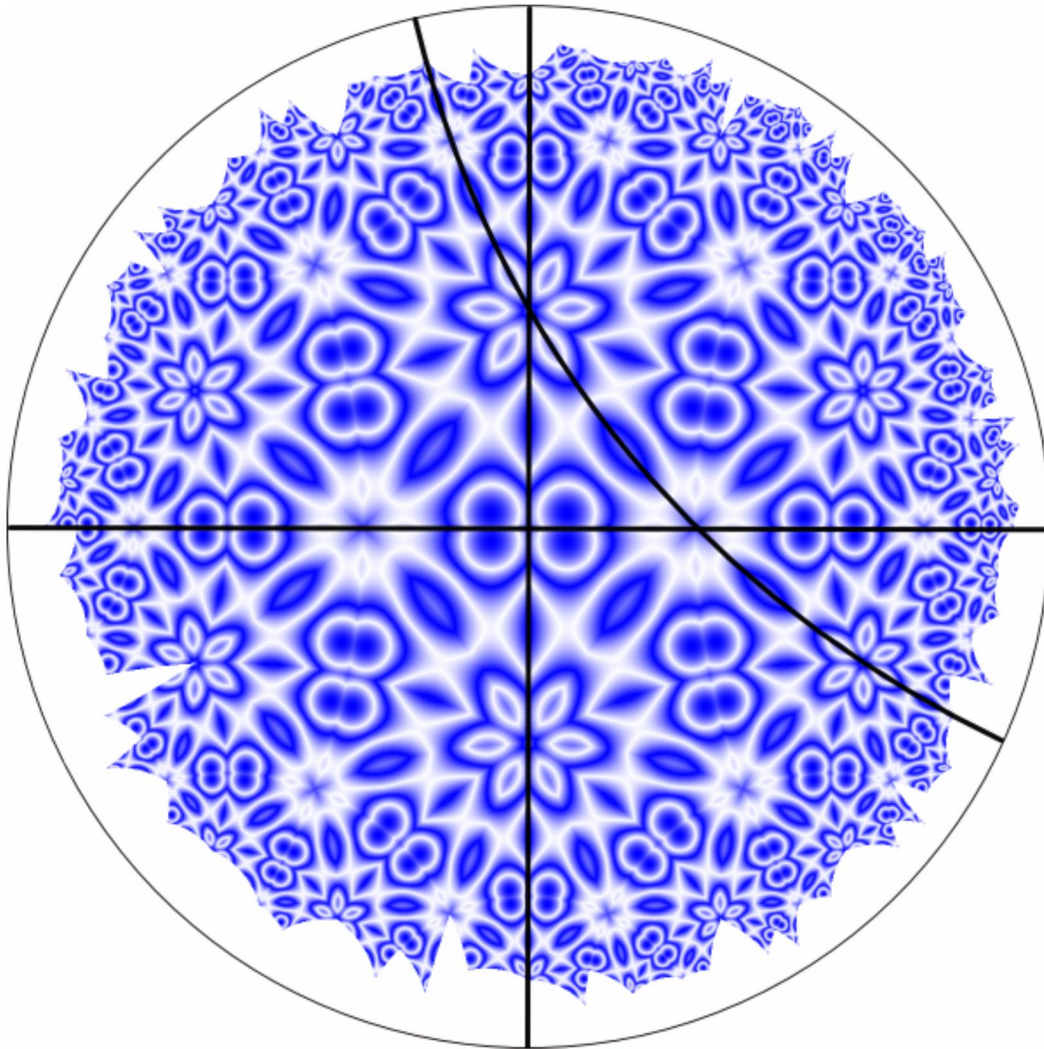
Hyperbolic triangle kaleidoscope

Make 8 reflections



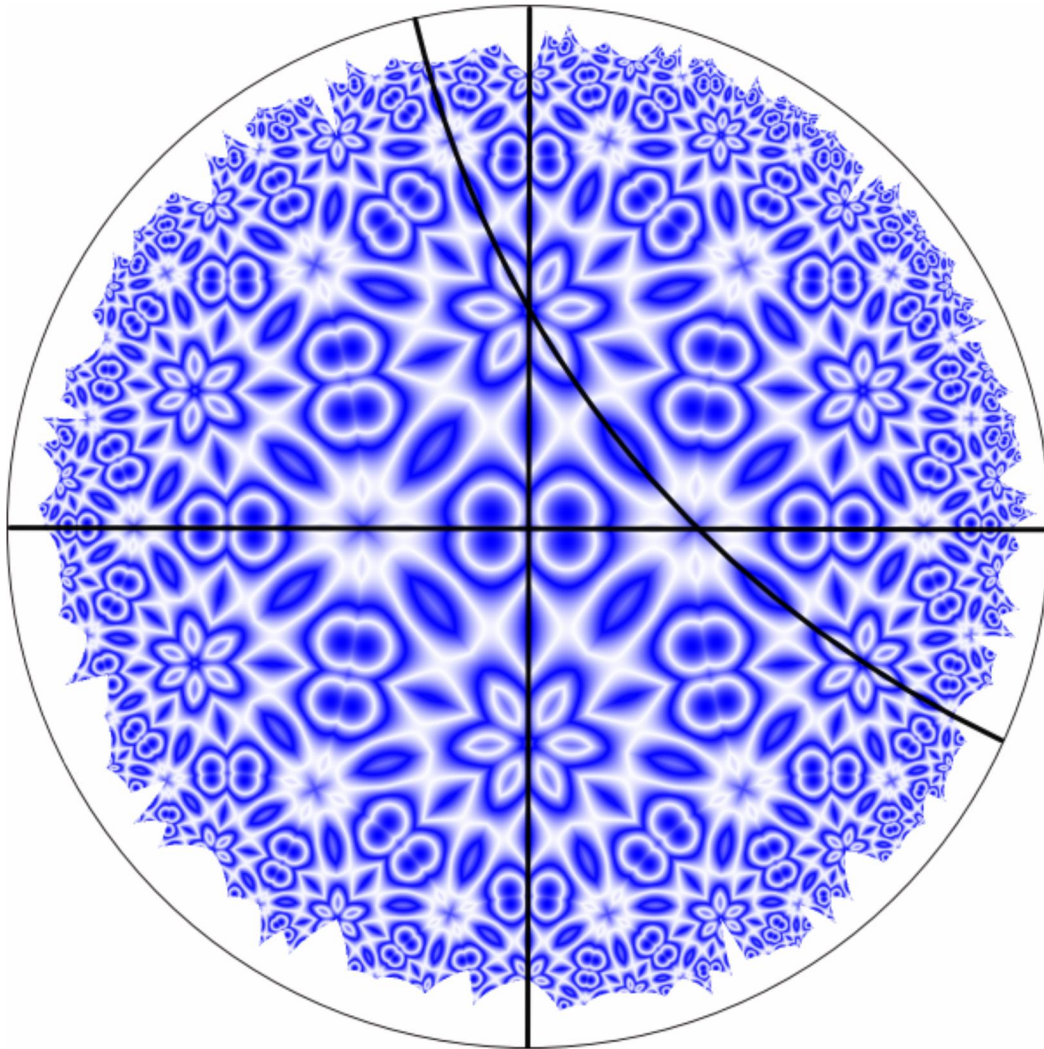
Hyperbolic triangle kaleidoscope

Make 9 reflections



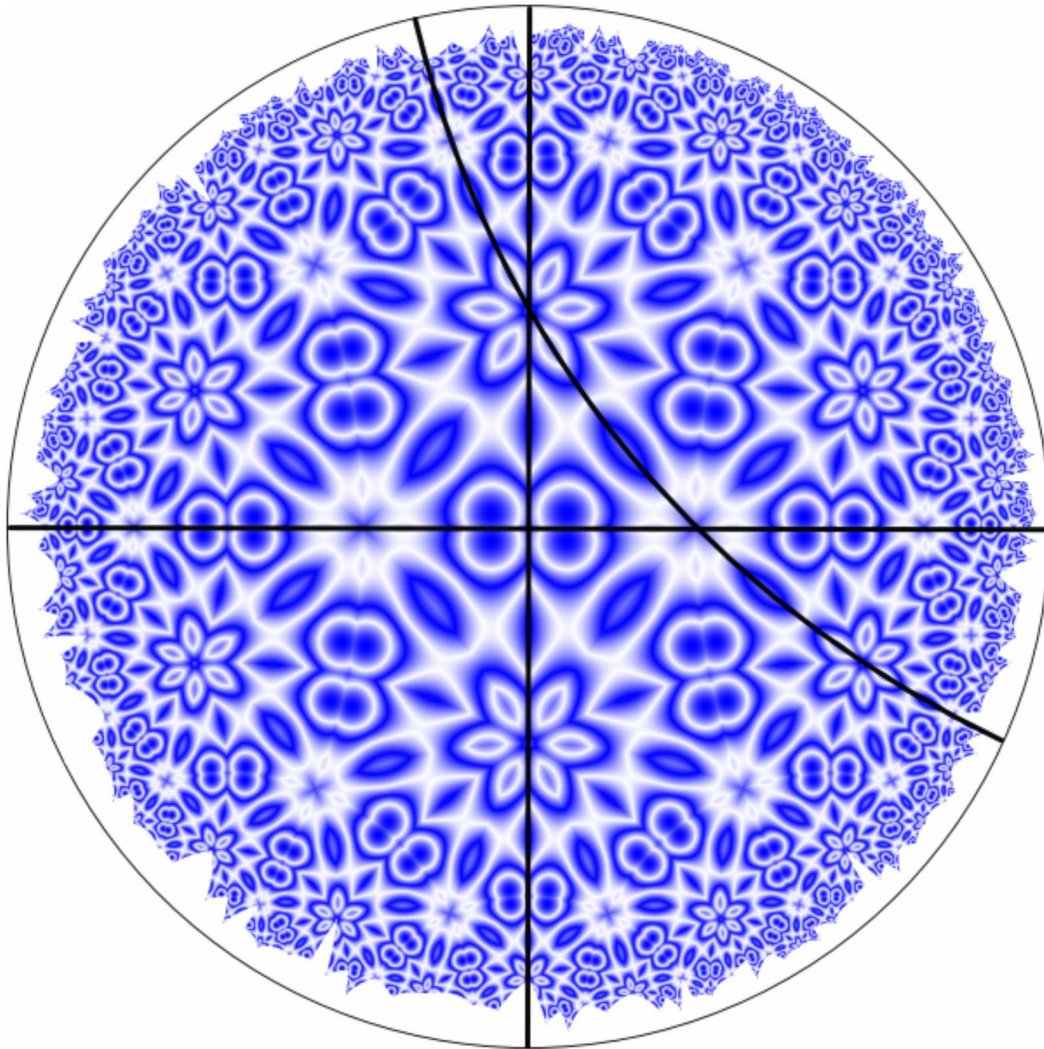
Hyperbolic triangle kaleidoscope

Make 10 reflections



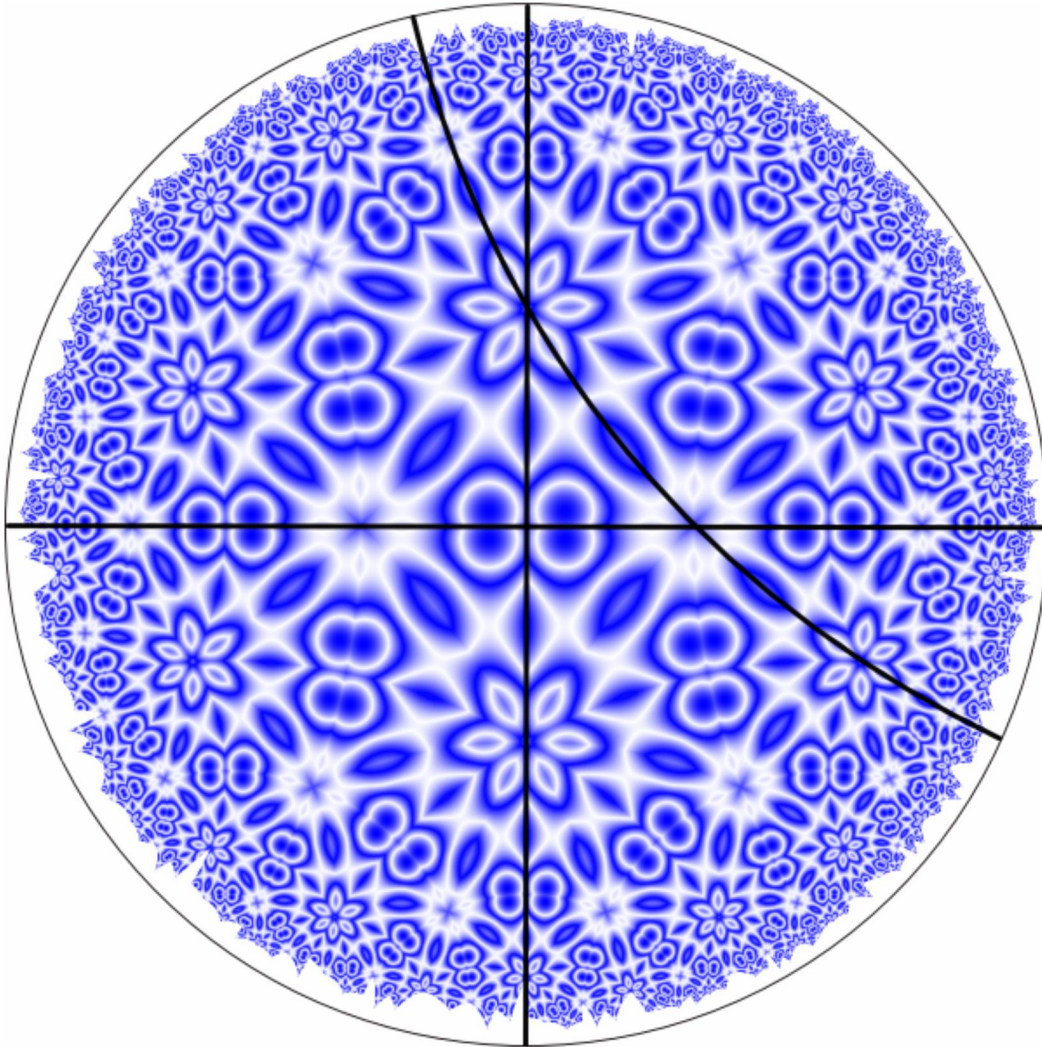
Hyperbolic triangle kaleidoscope

Make 11 reflections



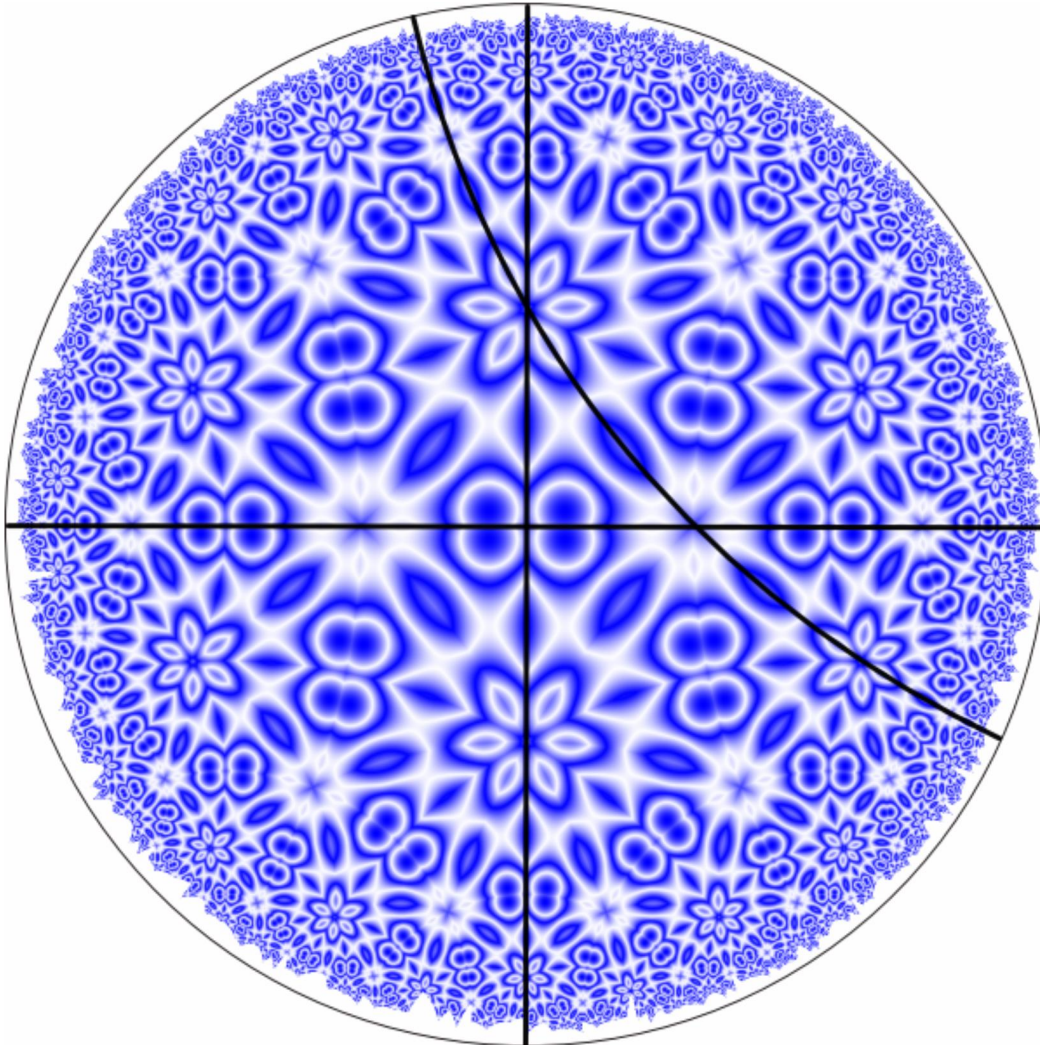
Hyperbolic triangle kaleidoscope

Make 12 reflections



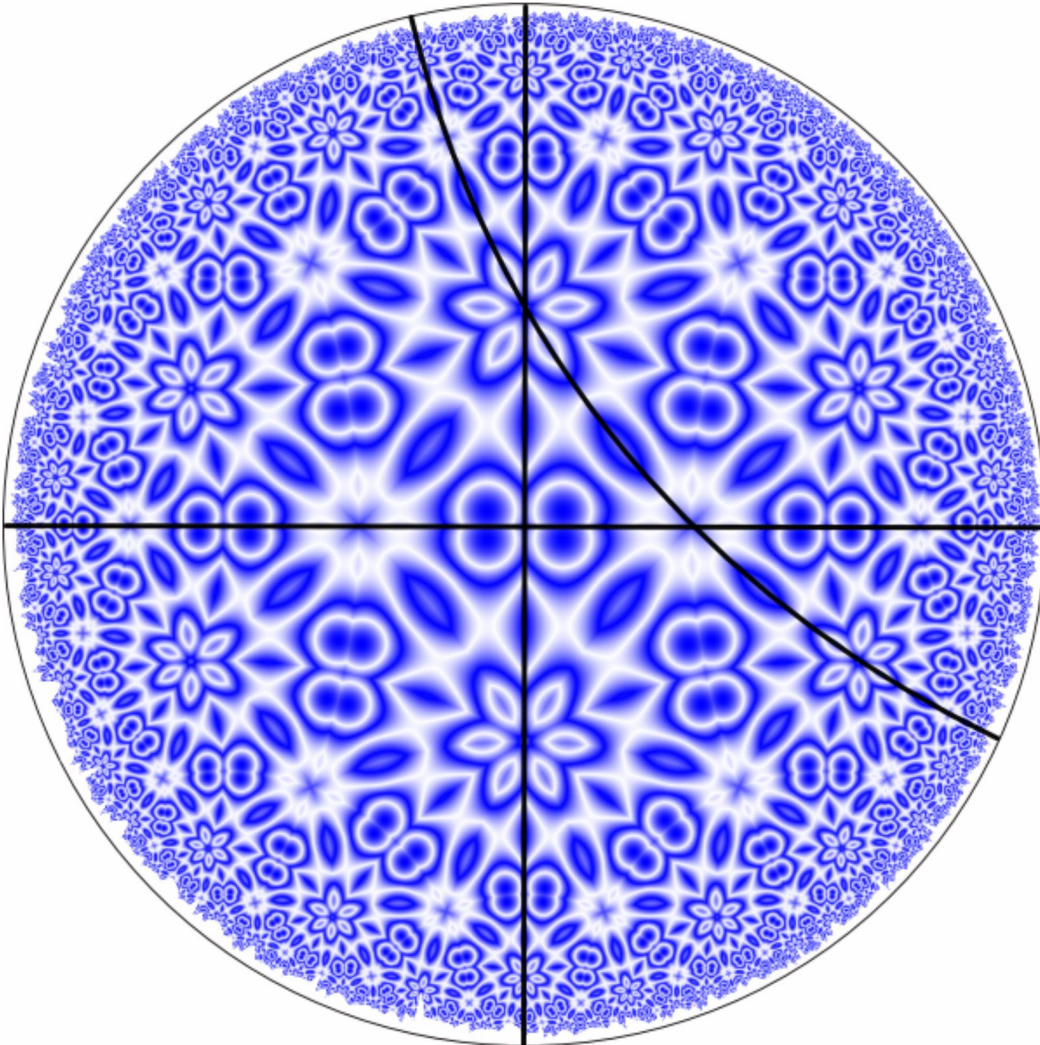
Hyperbolic triangle kaleidoscope

Make 13 reflections



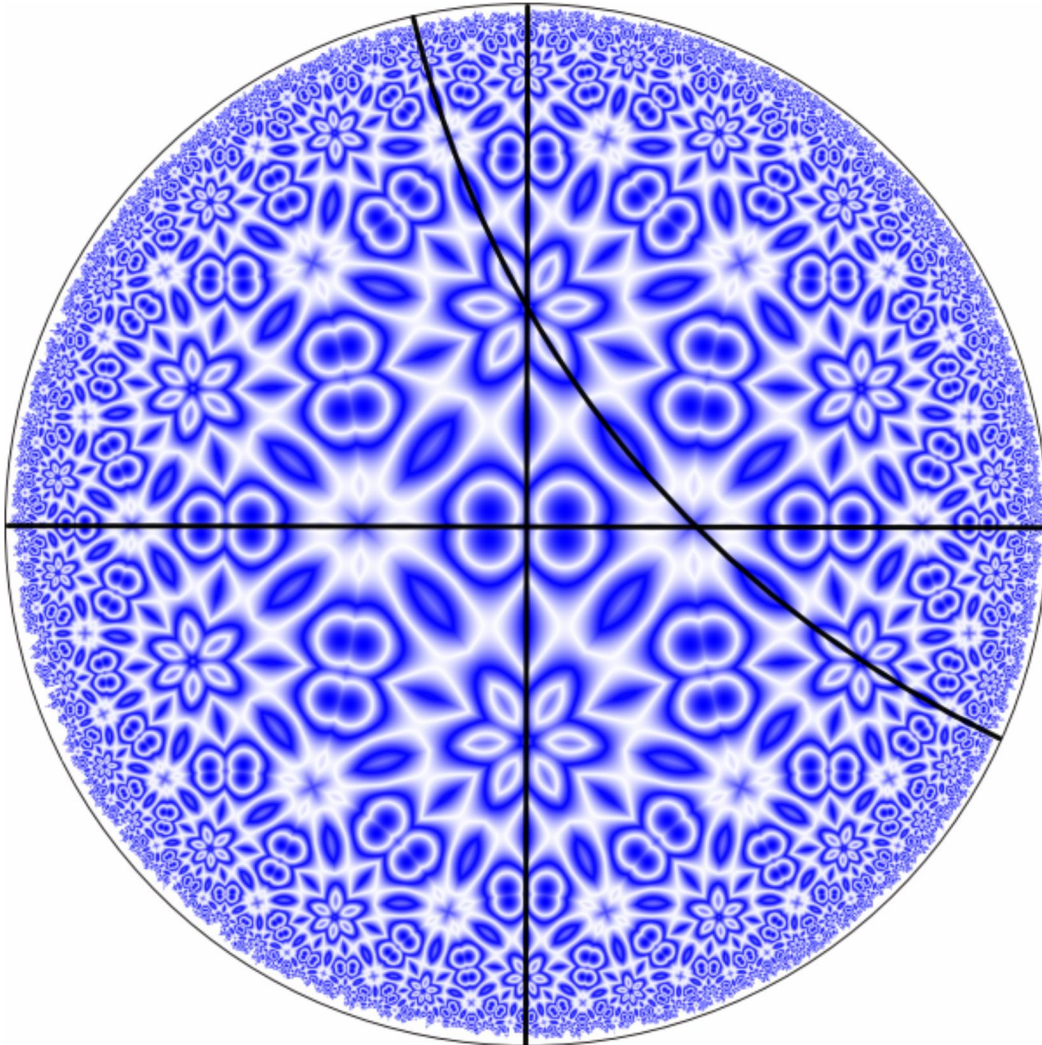
Hyperbolic triangle kaleidoscope

Make 14 reflections



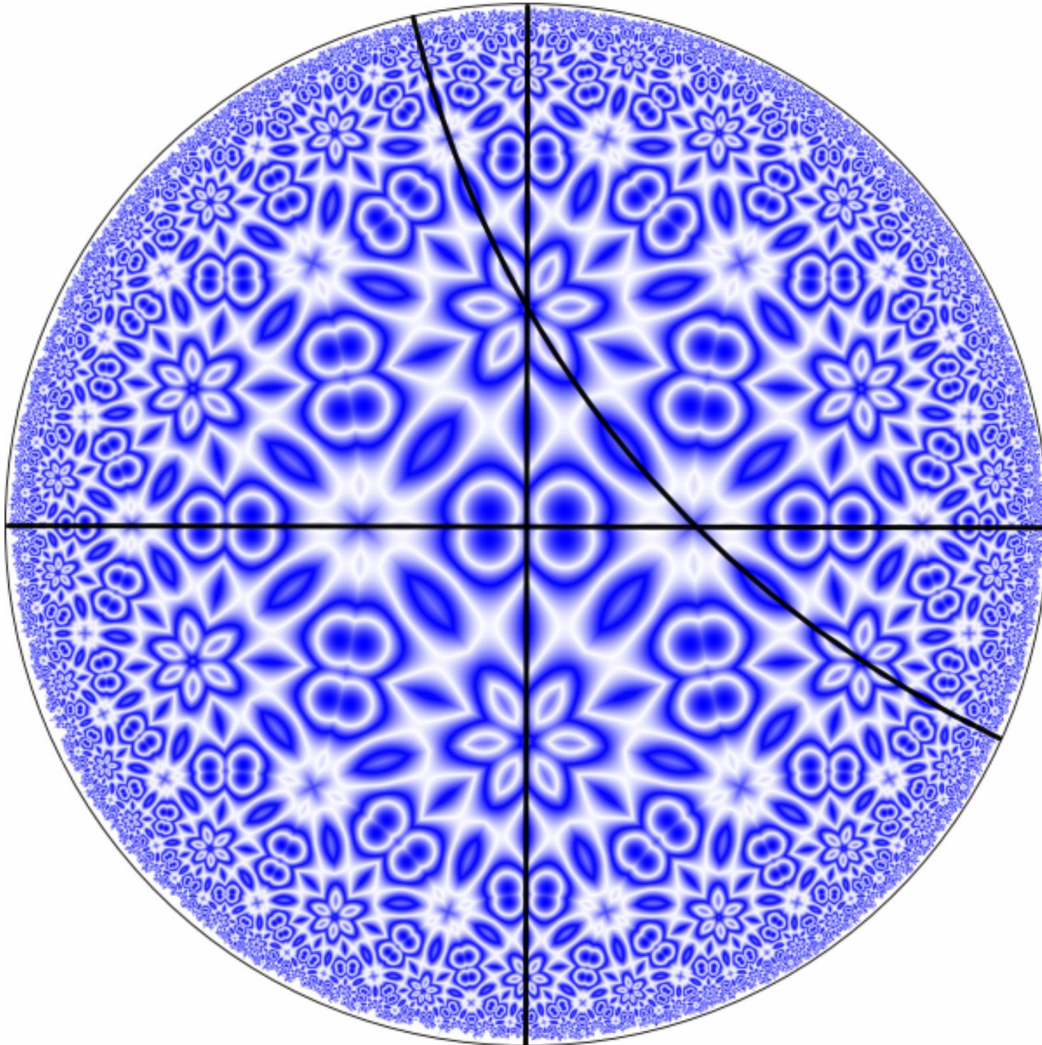
Hyperbolic triangle kaleidoscope

Make 15 reflections



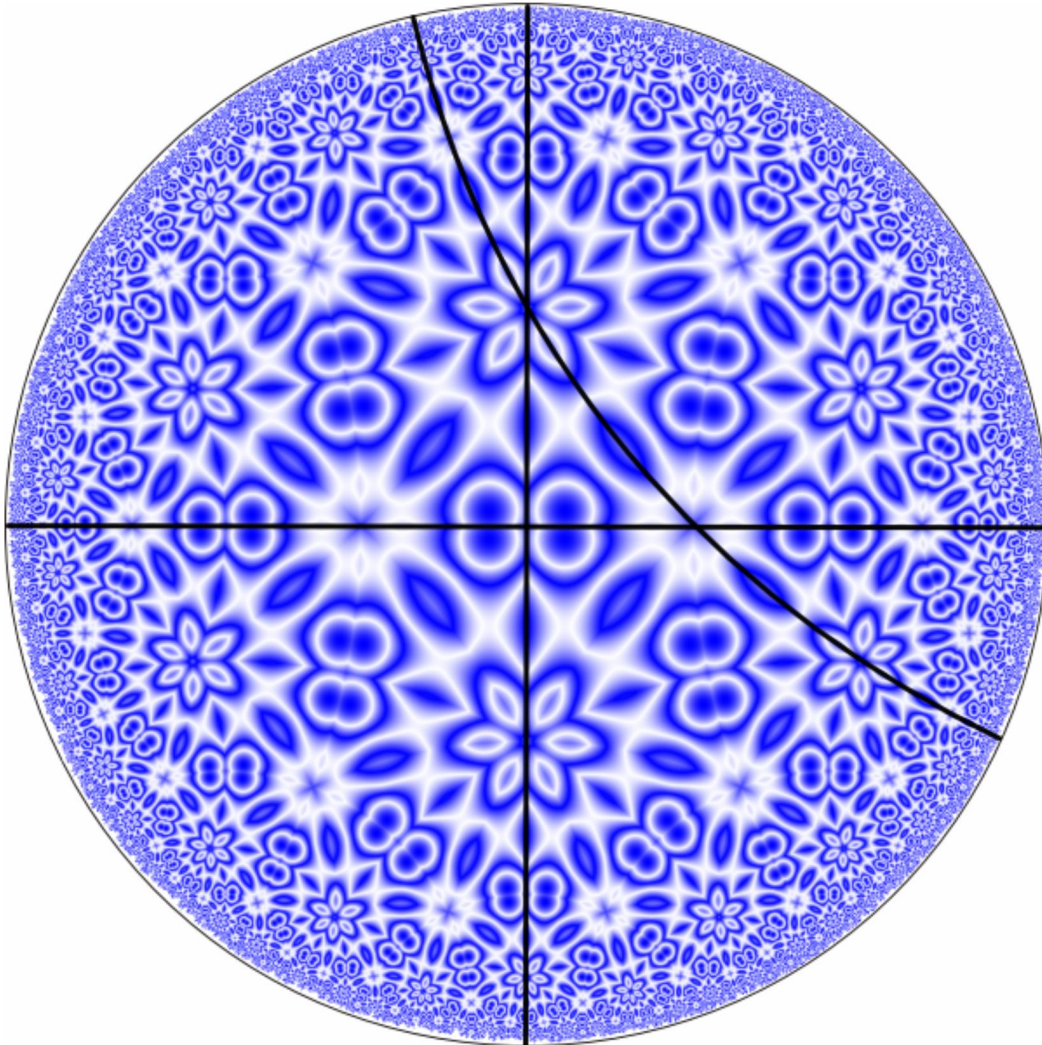
Hyperbolic triangle kaleidoscope

Make 16 reflections



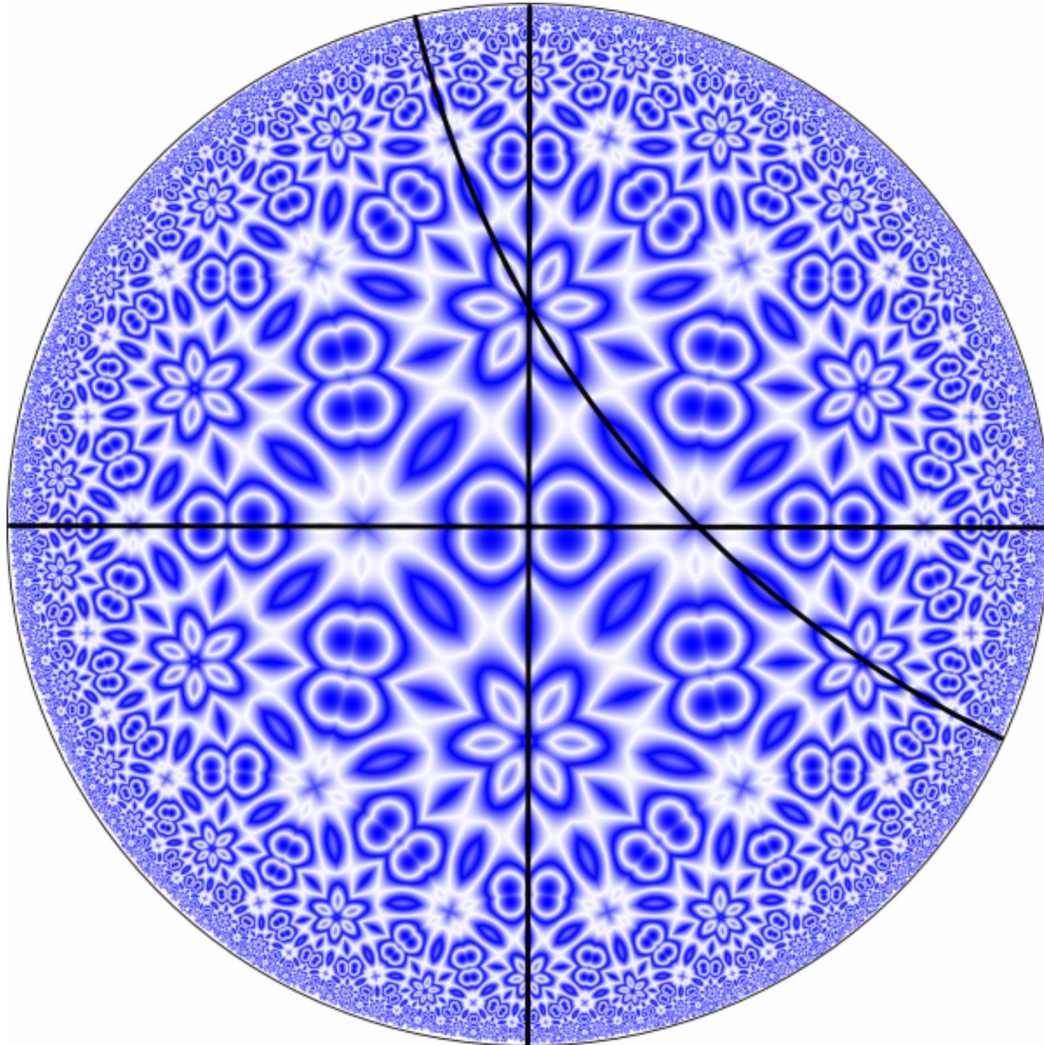
Hyperbolic triangle kaleidoscope

Make 17 reflections



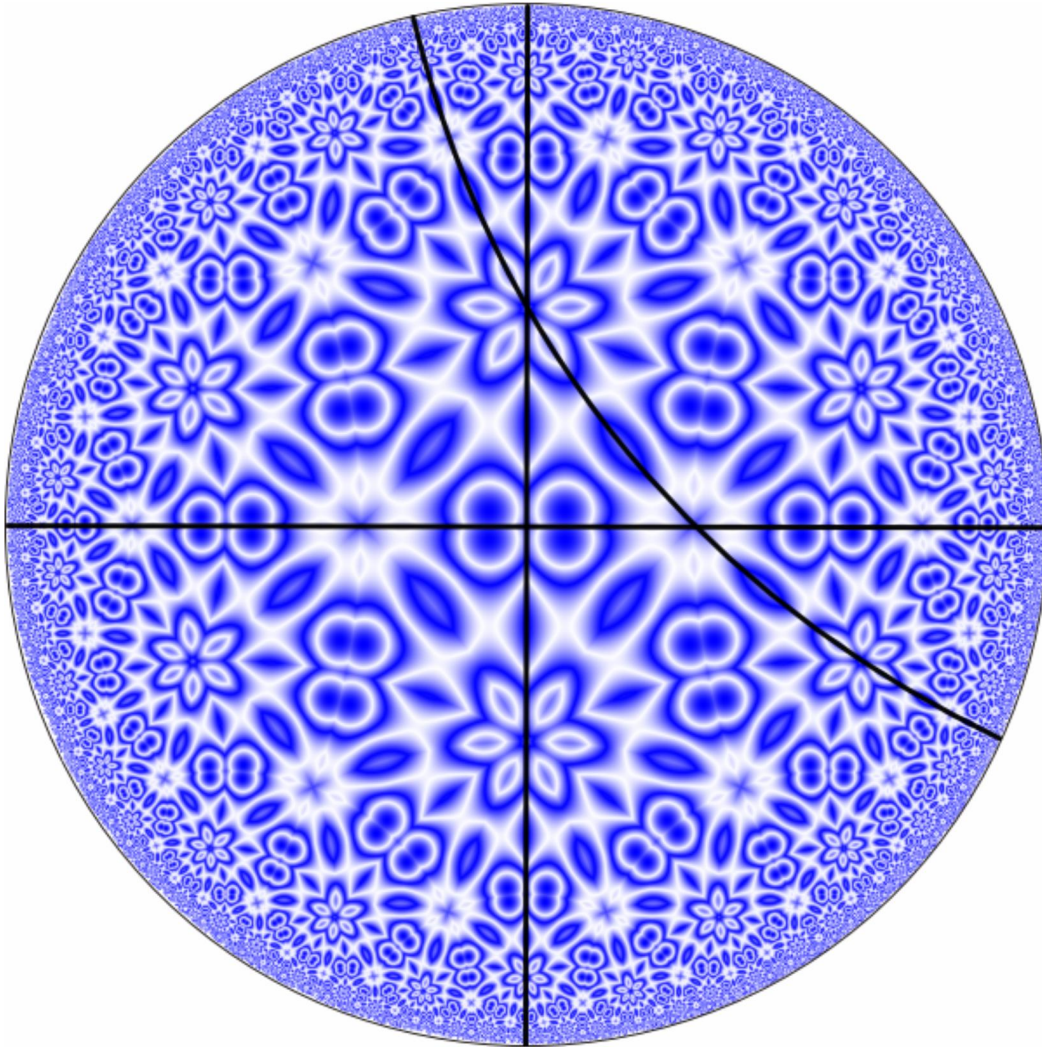
Hyperbolic triangle kaleidoscope

Make 18 reflections



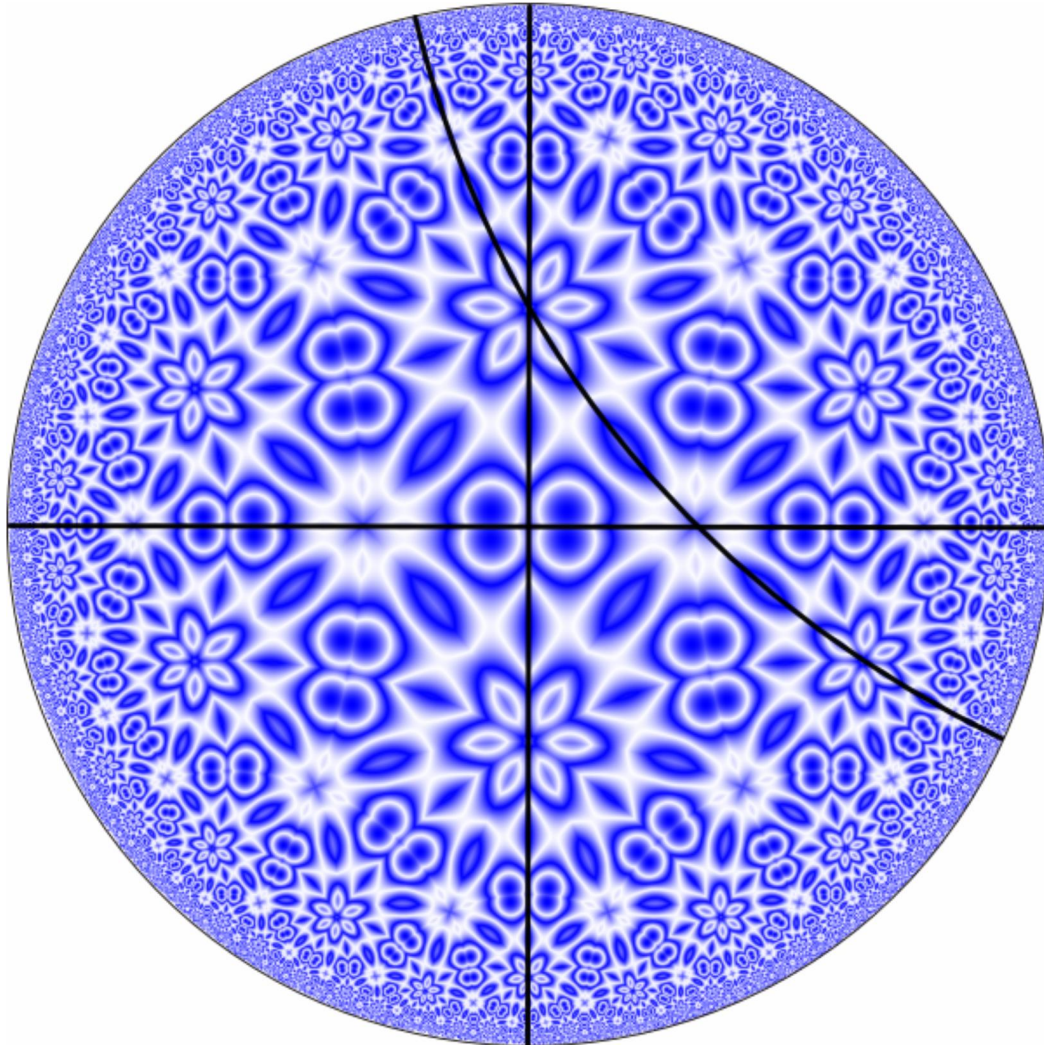
Hyperbolic triangle kaleidoscope

Make 19 reflections

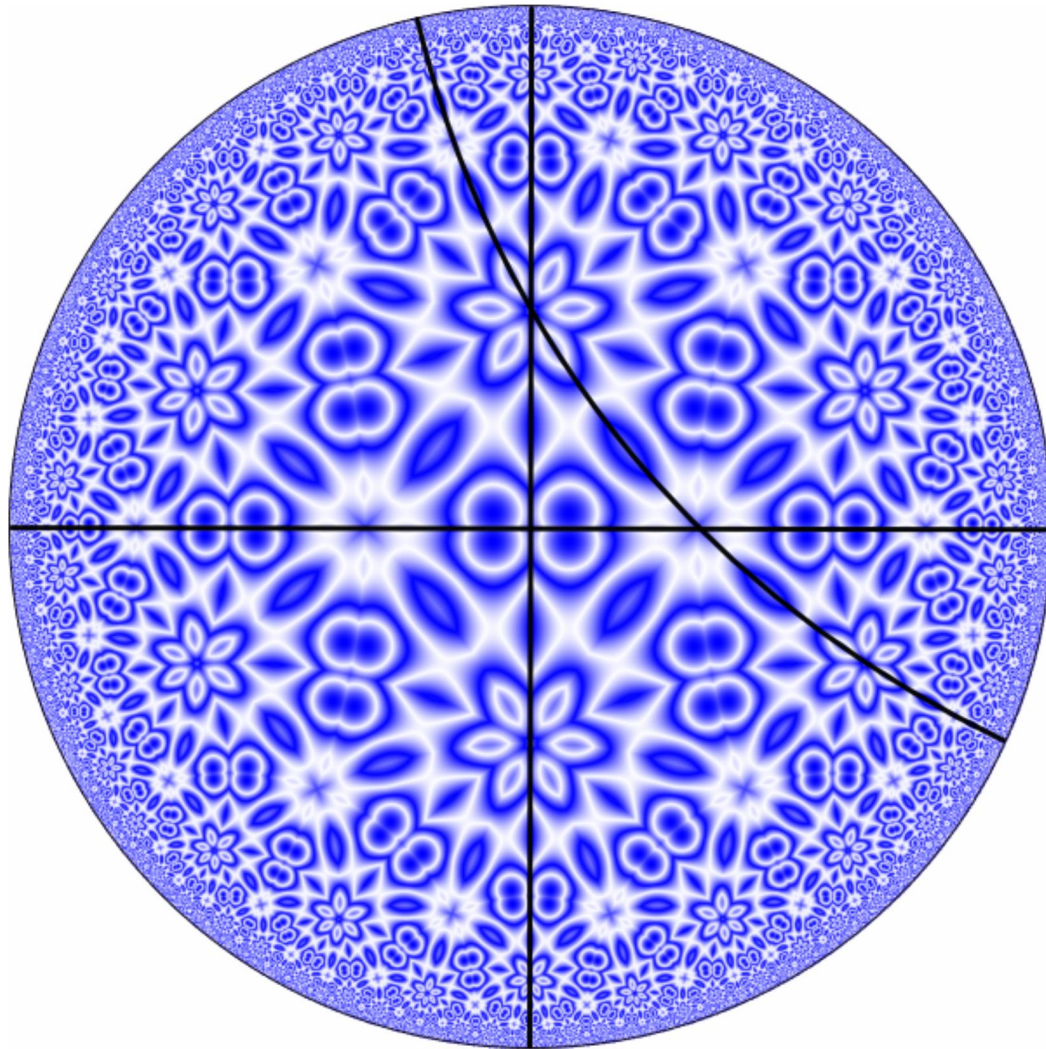


Hyperbolic triangle kaleidoscope

Make 20 reflections

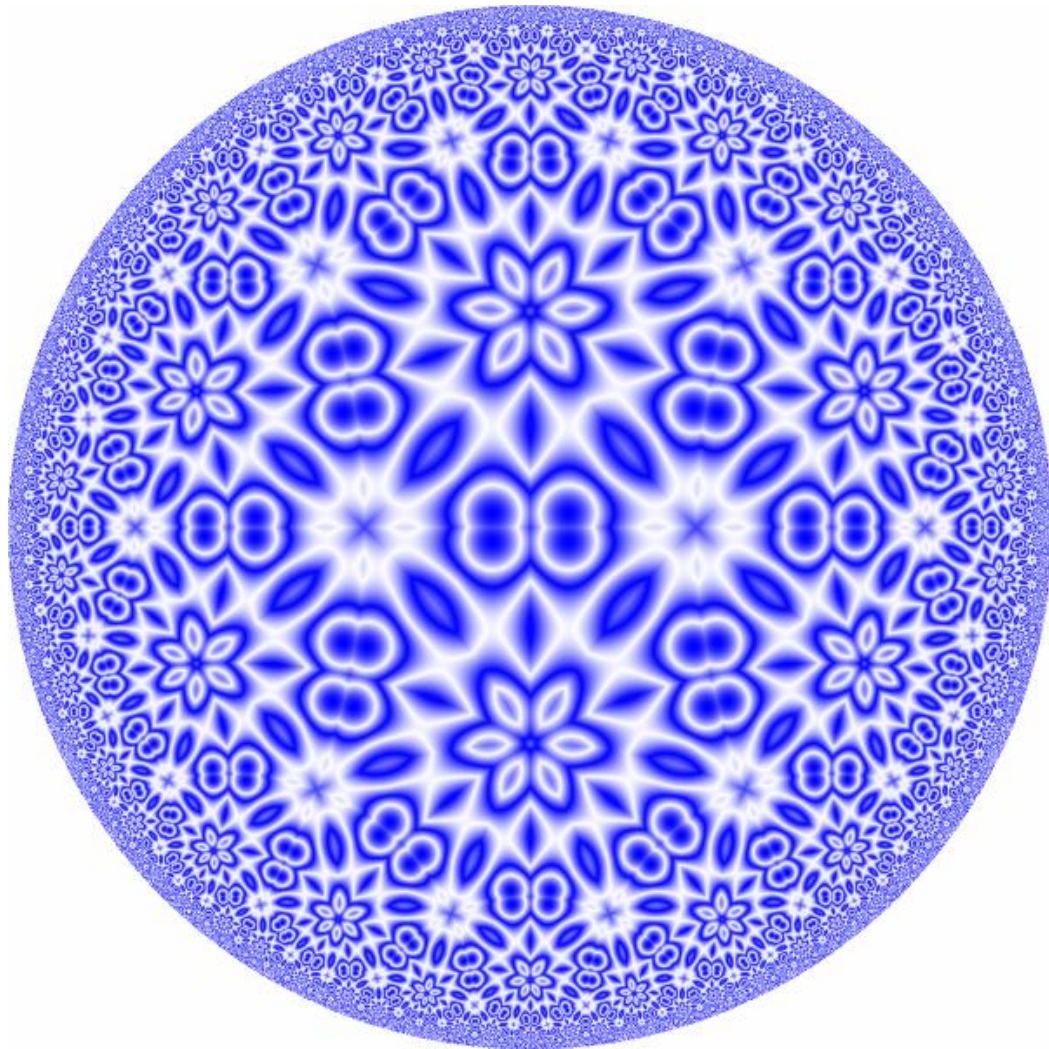


Hyperbolic triangle kaleidoscope



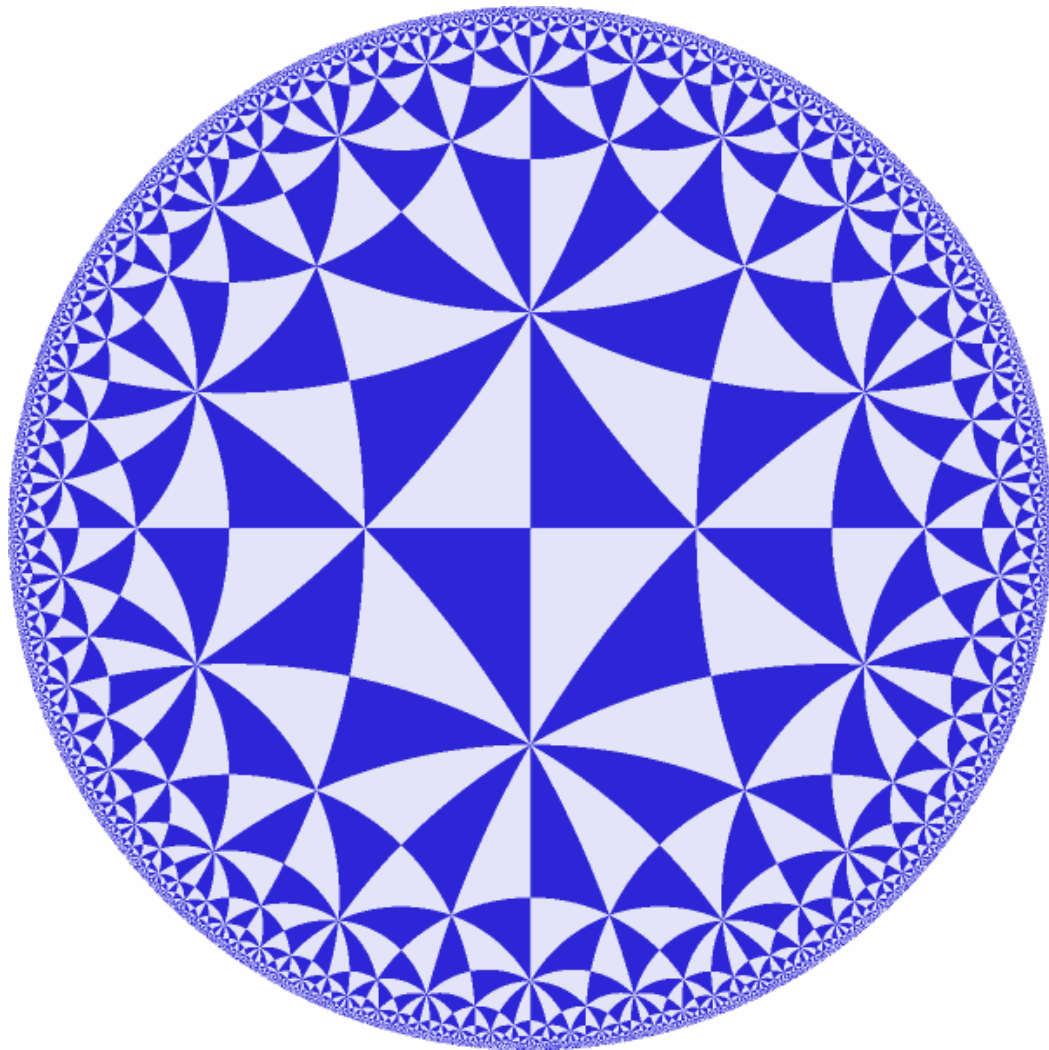
"Infinite" number of reflections.

Hyperbolic triangle kaleidoscope



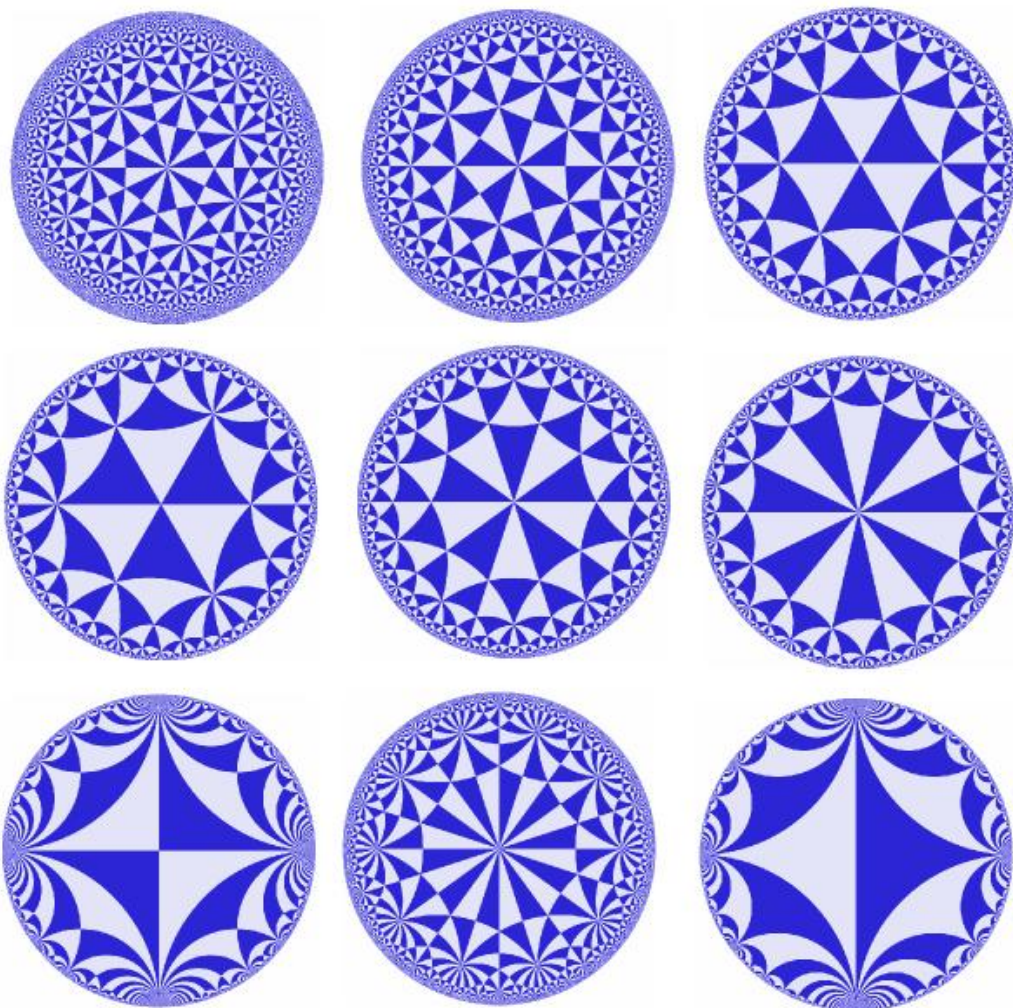
Complete hypebolc kaleidoscope *246.

Hyperbolic triangle kaleidoscope



Show different copies of fundamental domain in alternating colors for better view of the "grid" structure.

other hyperbolic triangles



There are countable number of different triangular kaleidoscopes in the hyperbolic plane.

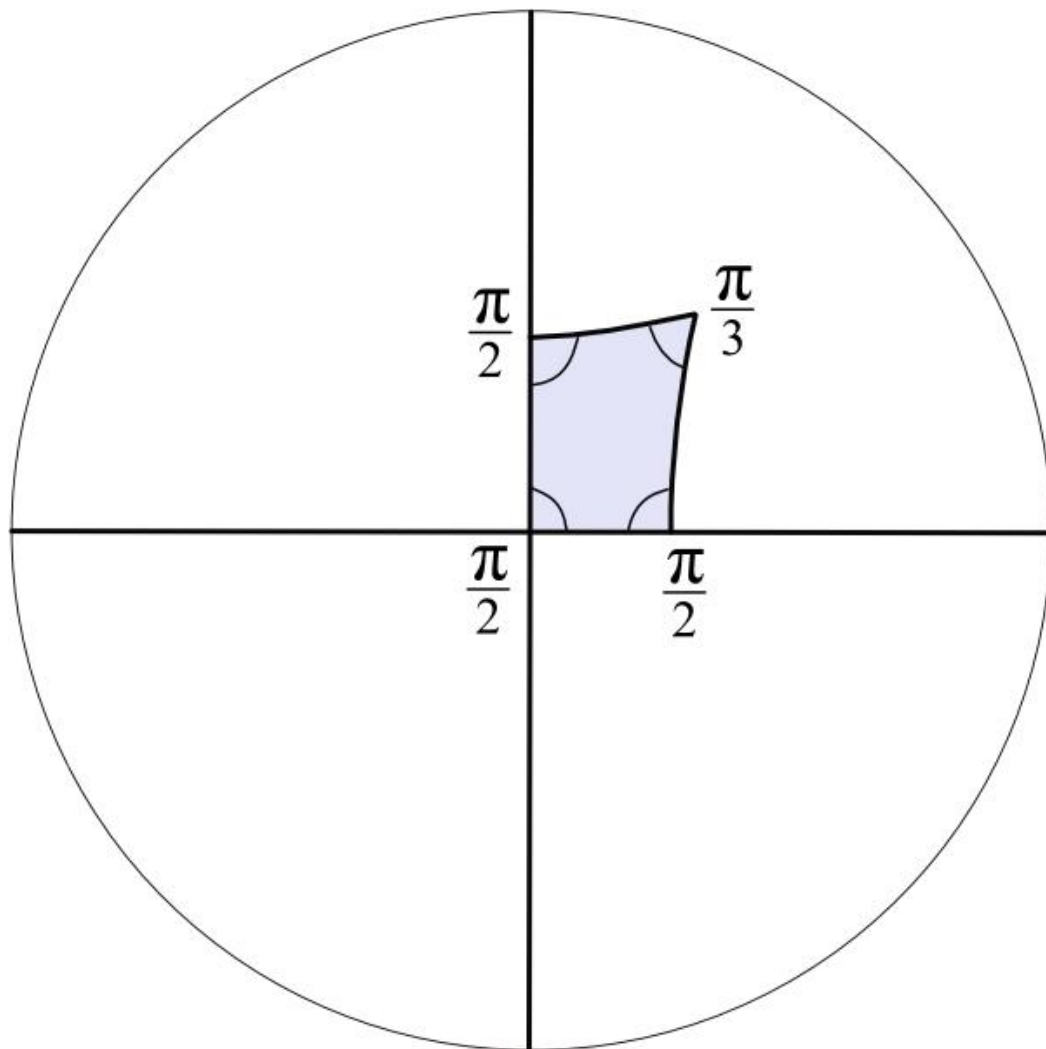
Angles of triangle should be submultiples of π or 0

$$\frac{\pi}{k}, \frac{\pi}{l}, \frac{\pi}{m}, l, m, n \geq 2$$

Sum should be less than π

$$\frac{\pi}{k} + \frac{\pi}{l} + \frac{\pi}{m} < \pi$$

Quadrilateral kaleidocscope

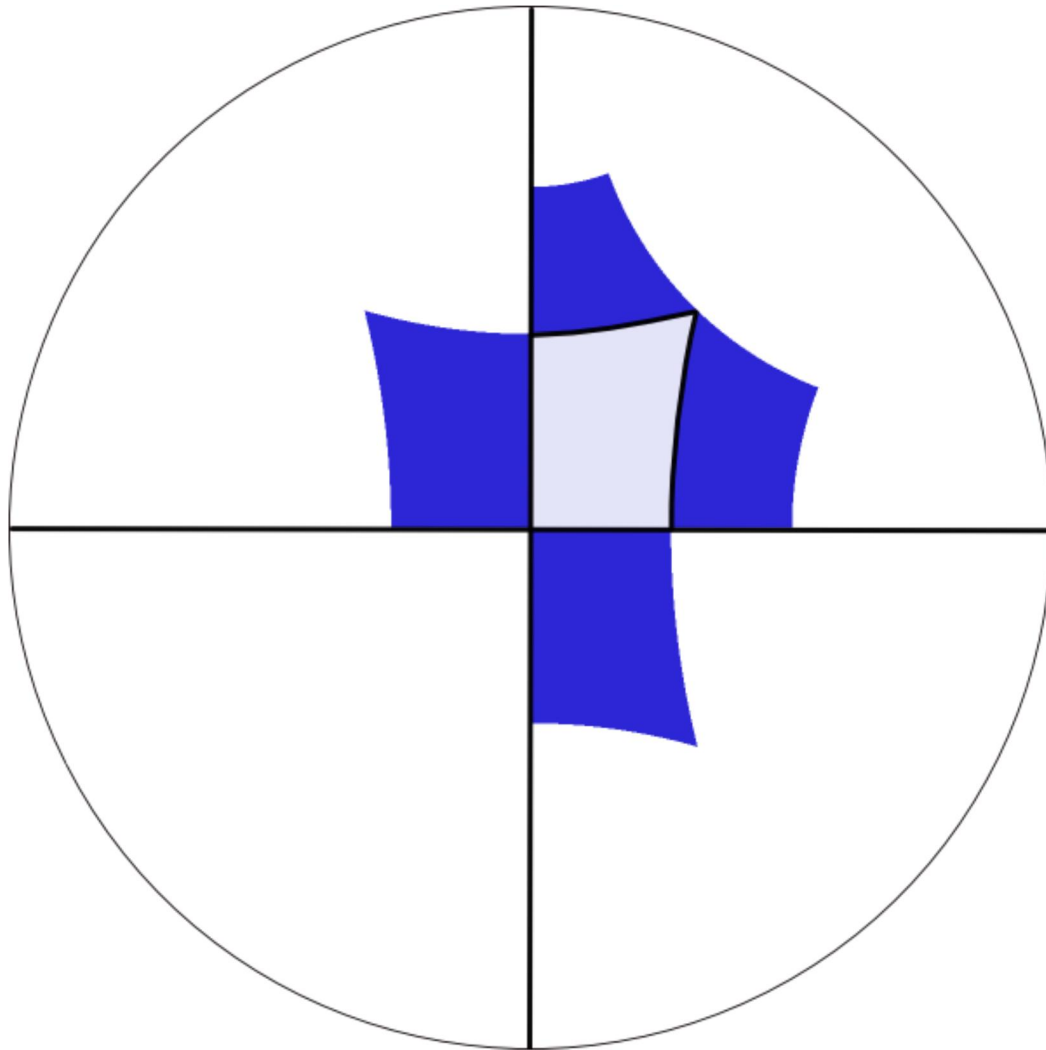


What about other polygons in hyperbolic plane?

Let's make quadrilateral with angles

$$\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}$$

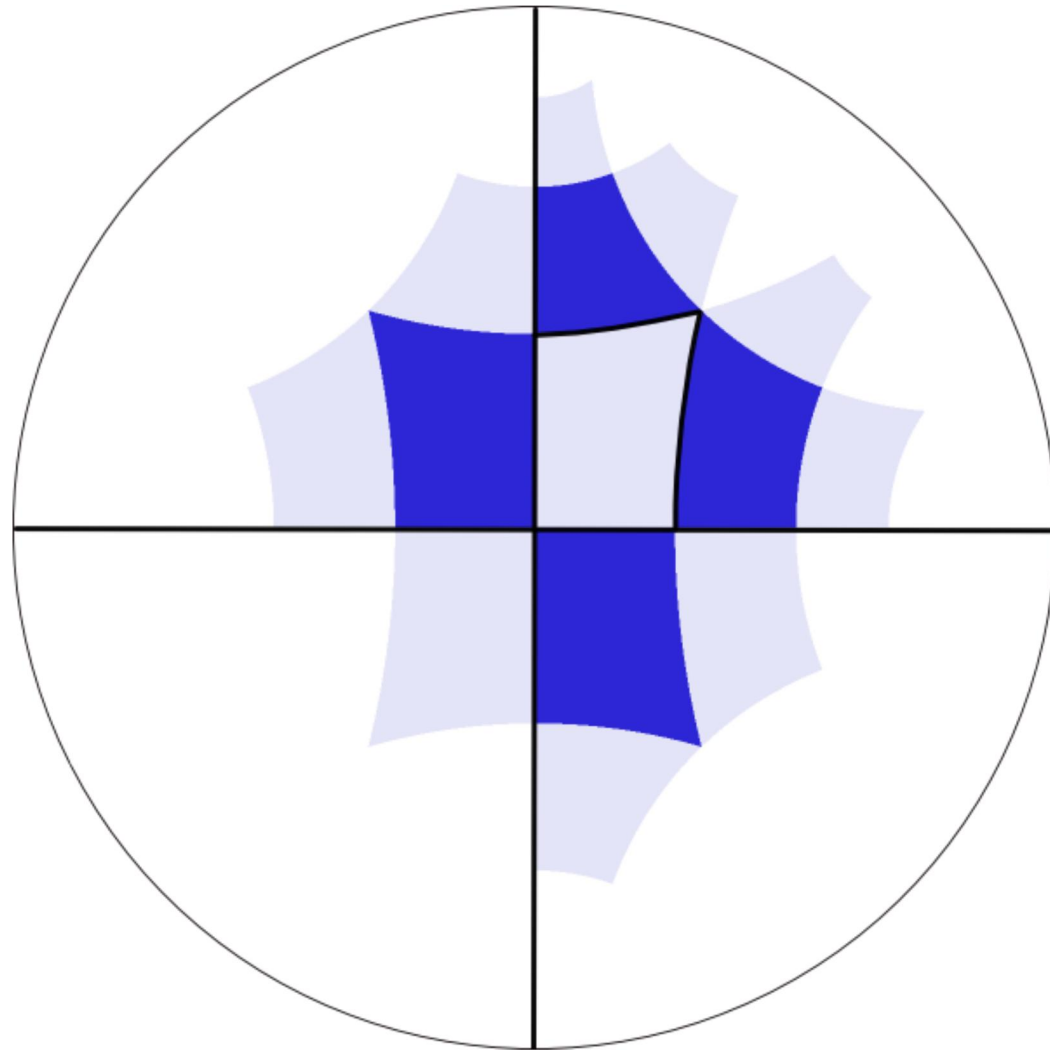
Quadrilateral kaleidocscope



Reflect in the sides of this quadrilateral.

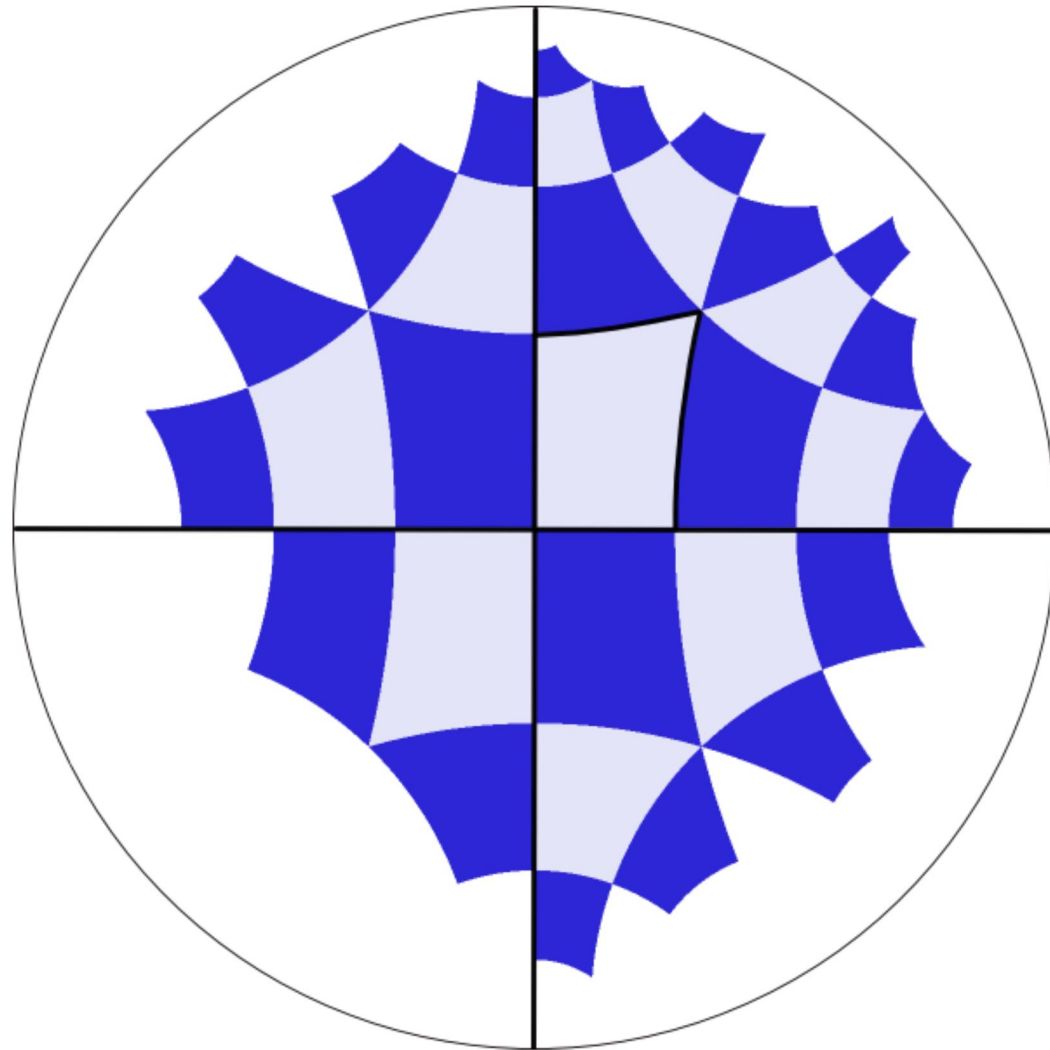
1 reflection

Quadrilateral kaleidocscope



2 reflections

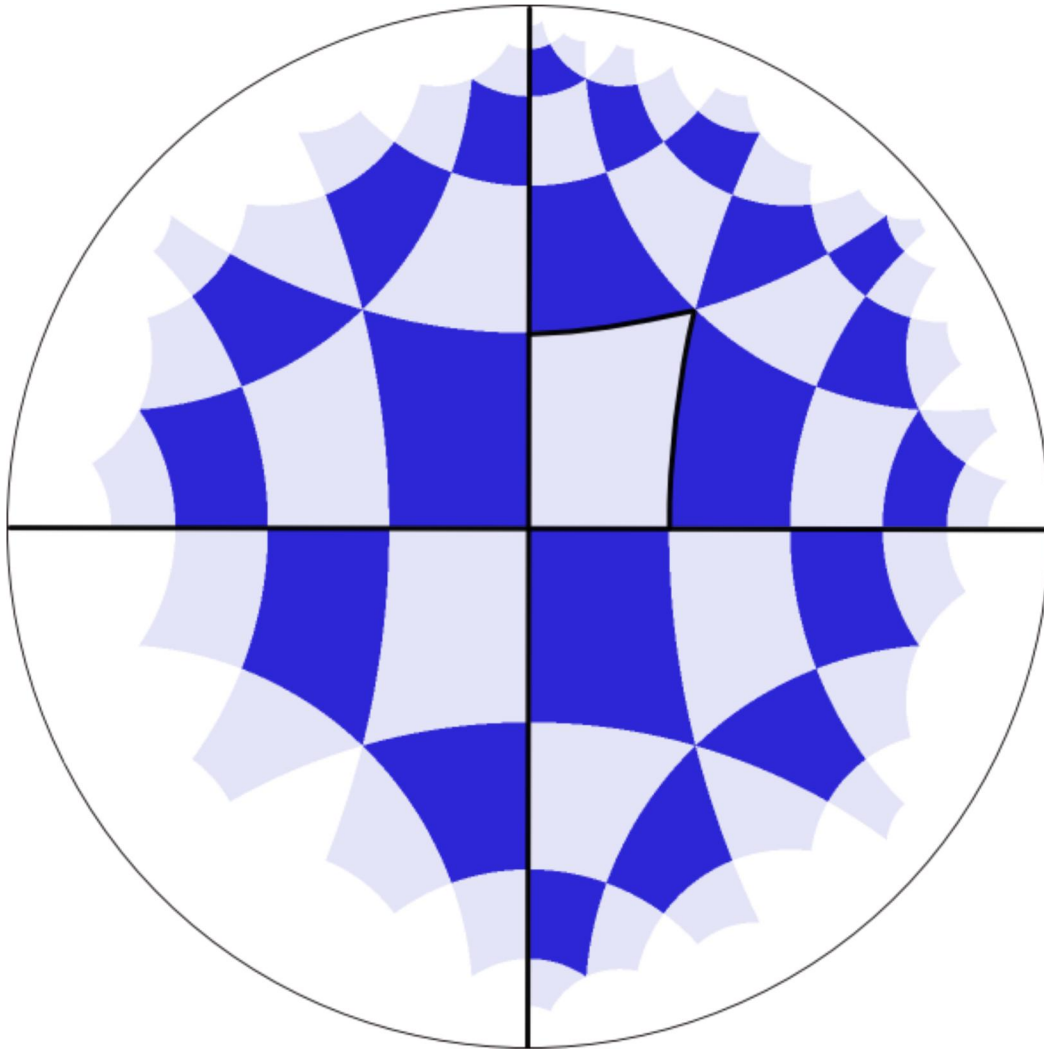
Quadrilateral kaleidocscope



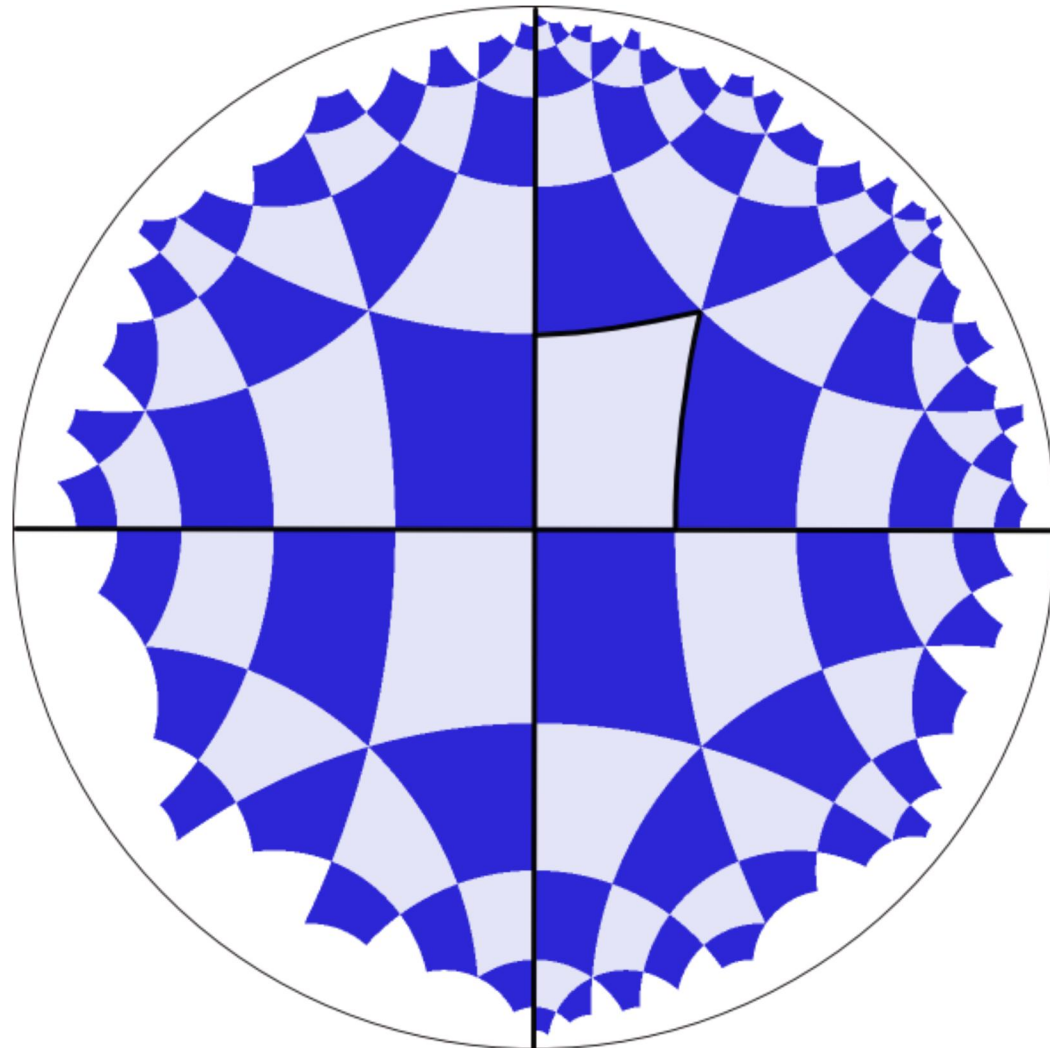
3 reflections

Quadrilateral kaleidocscope

4 reflections



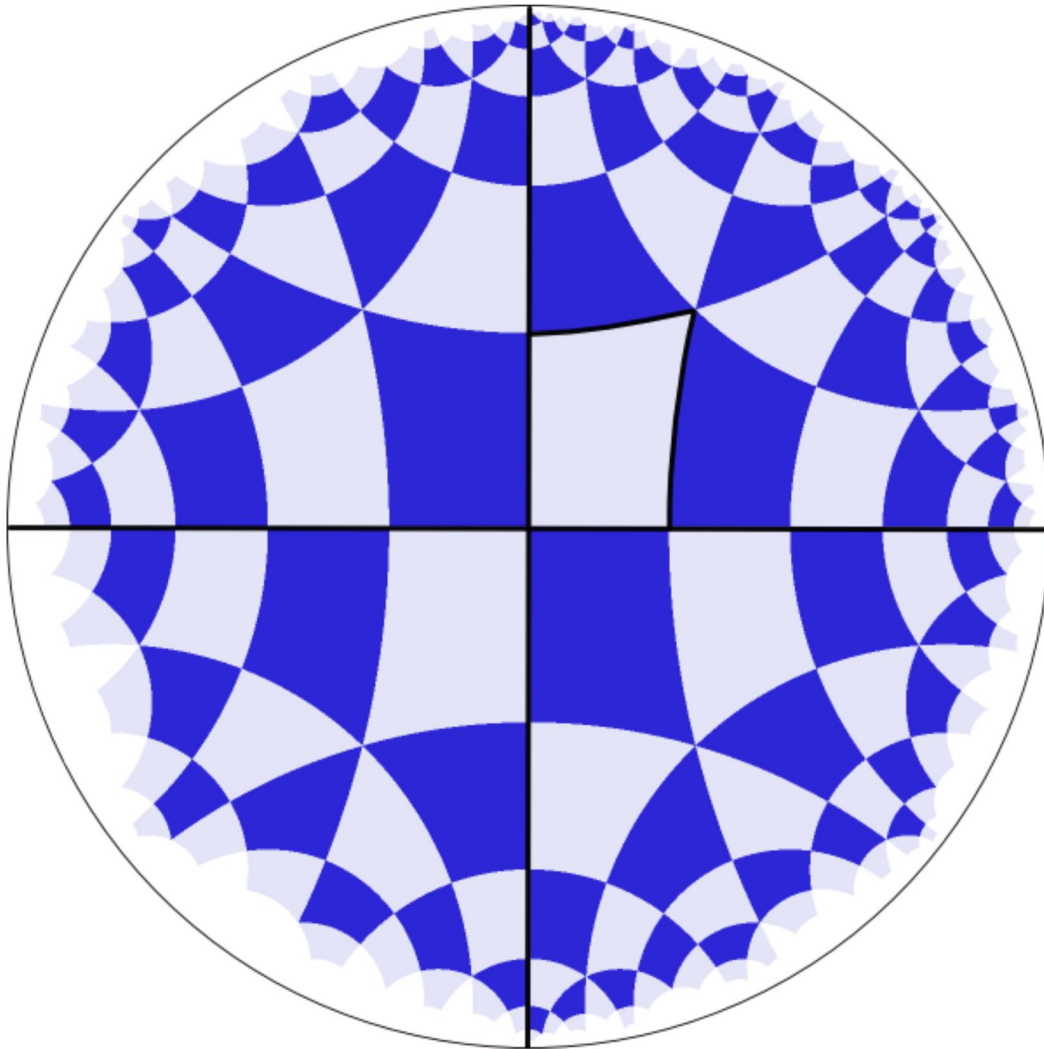
Quadrilateral kaleidocscope



5 reflections

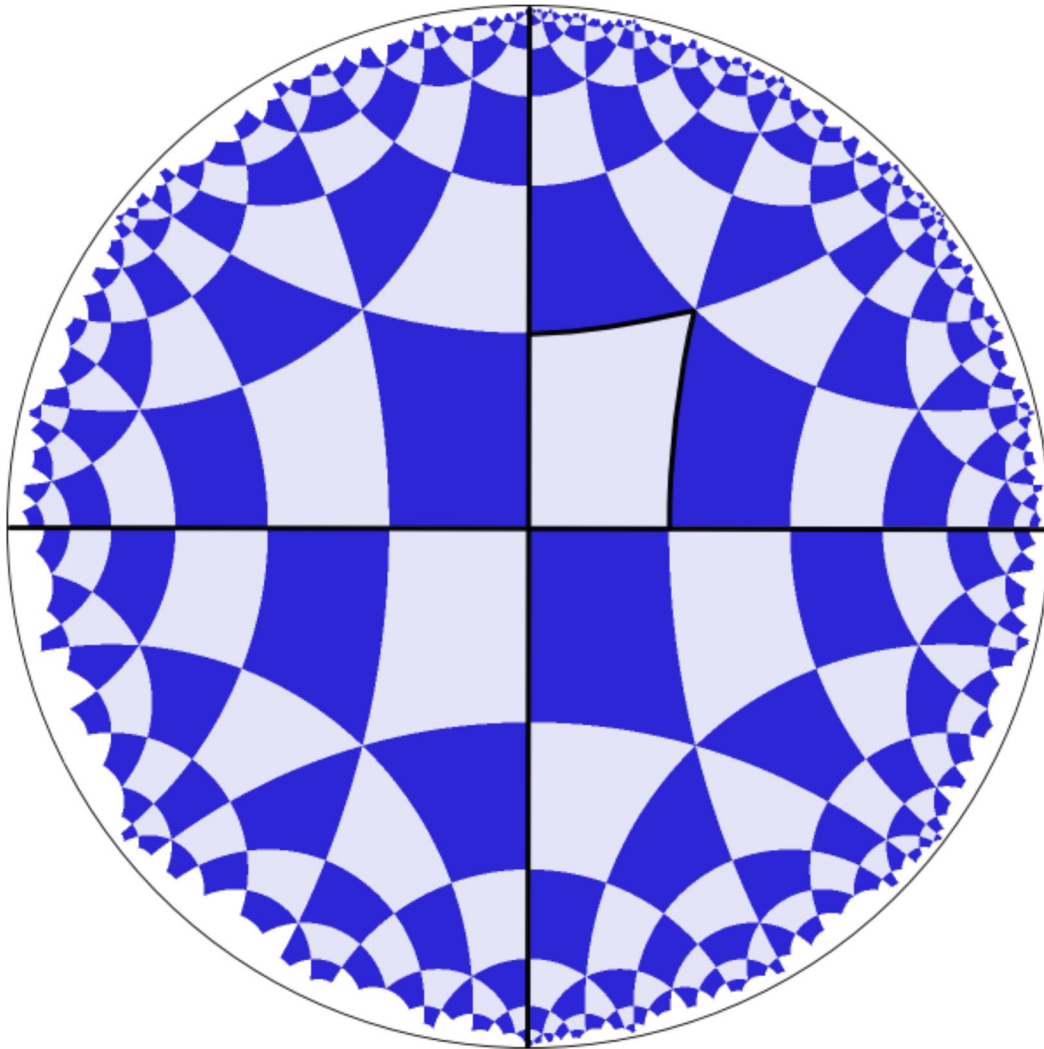
Quadrilateral kaleidocscope

6 reflections



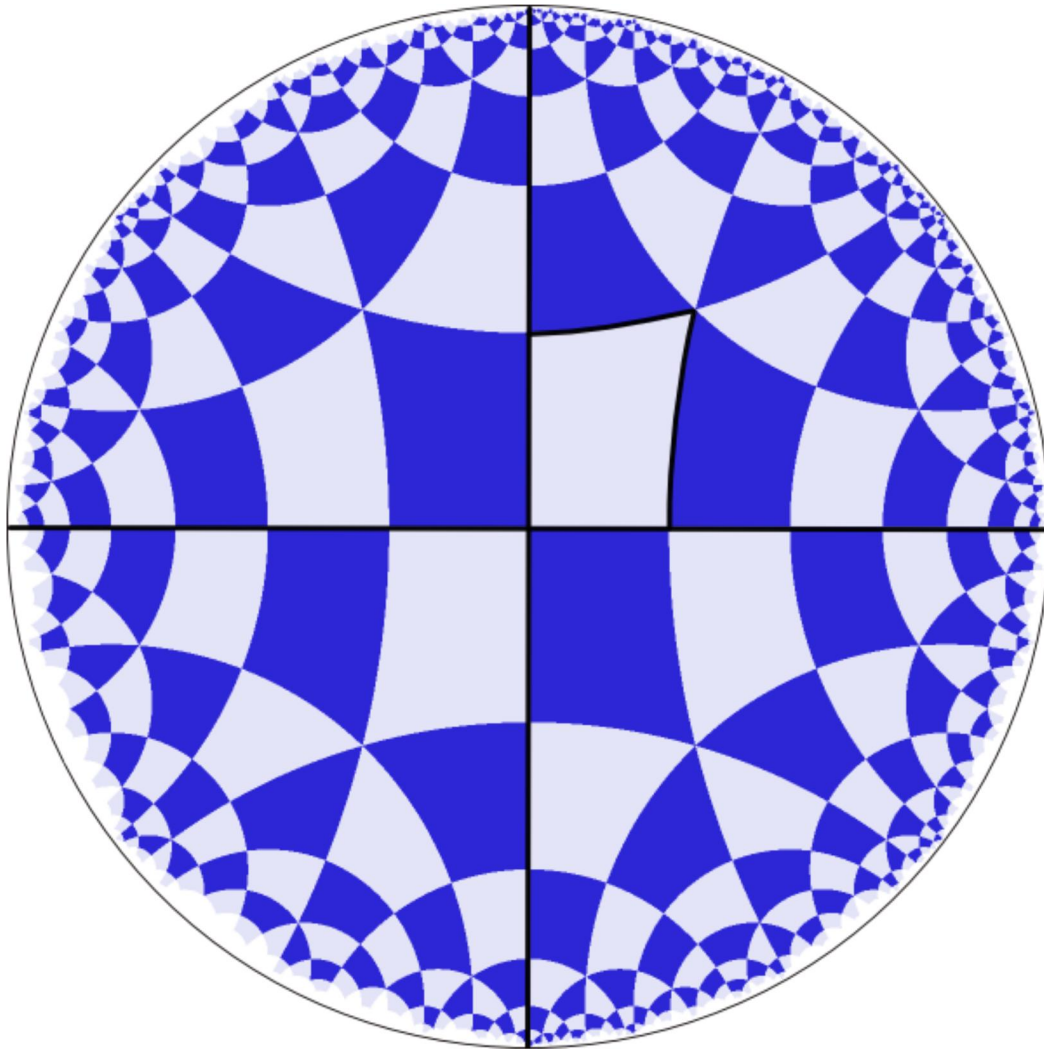
Quadrilateral kaleidocscope

7 reflections



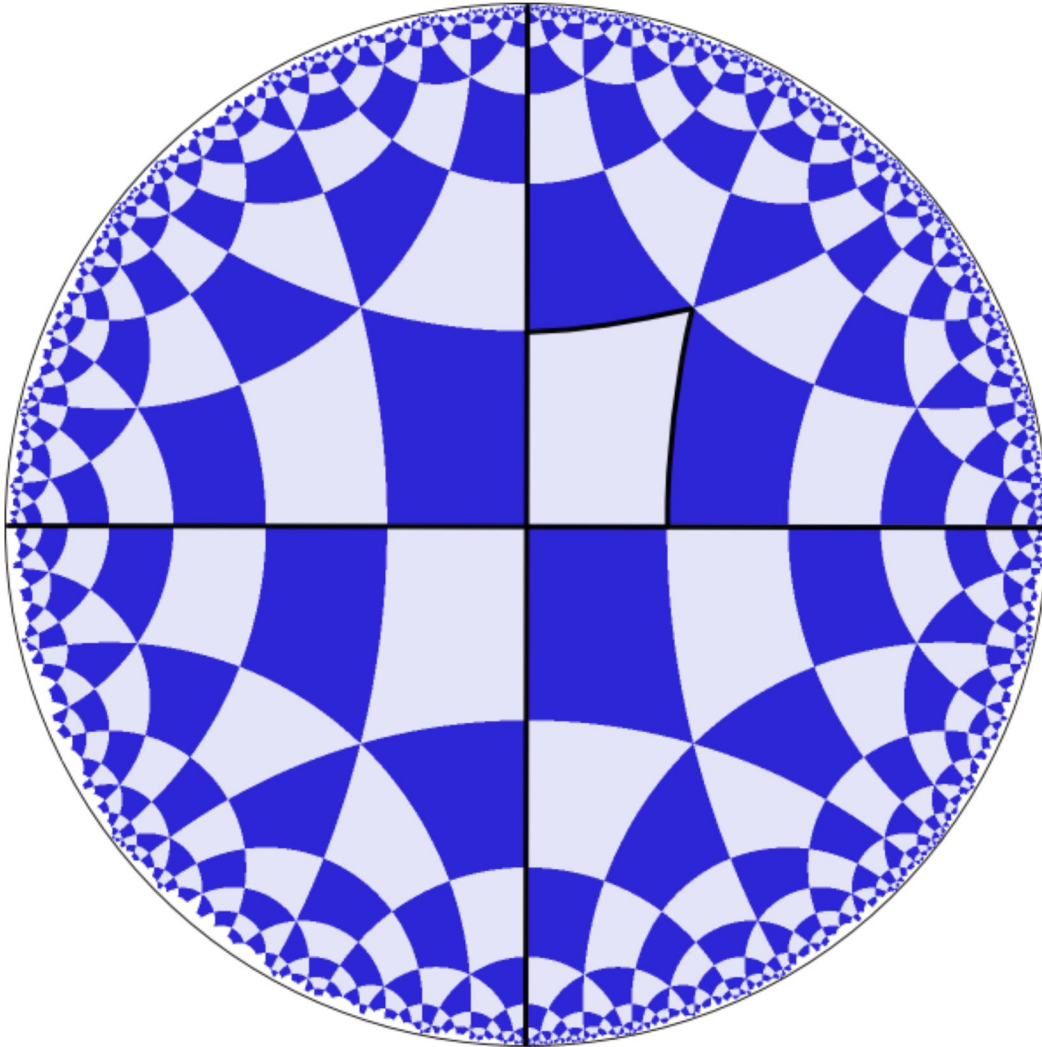
Quadrilateral kaleidocscope

8 reflections



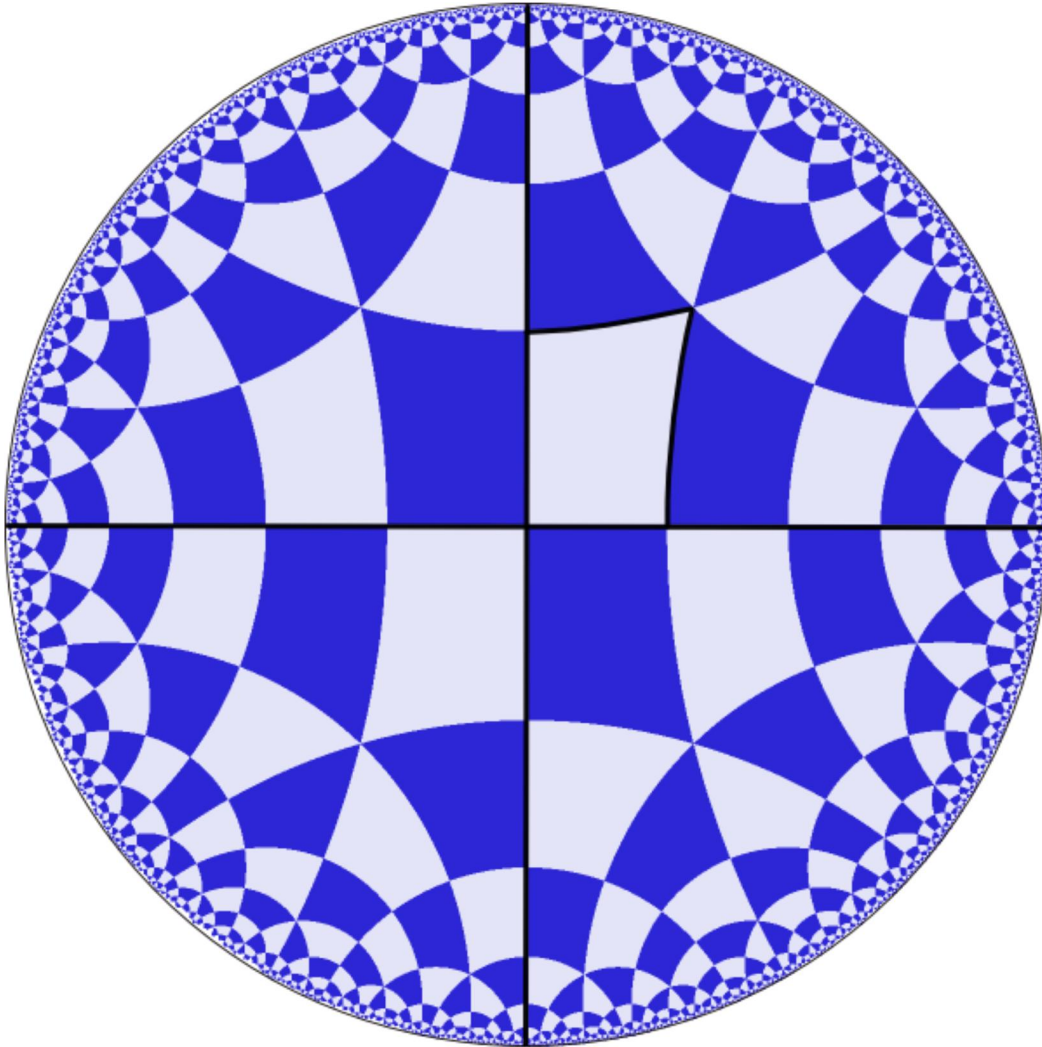
Quadrilateral kaleidocscope

9 reflections



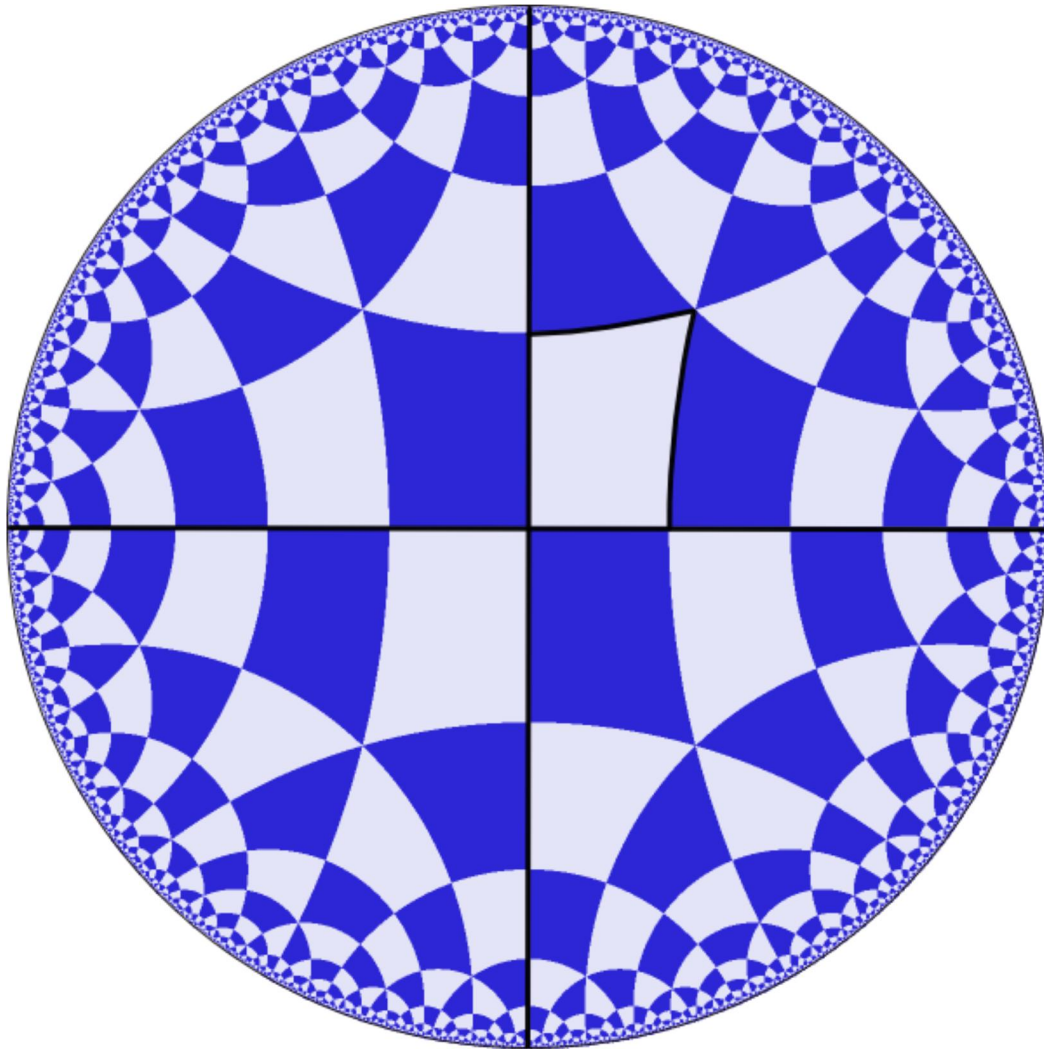
Quadrilateral kaleidocscope

11 reflections



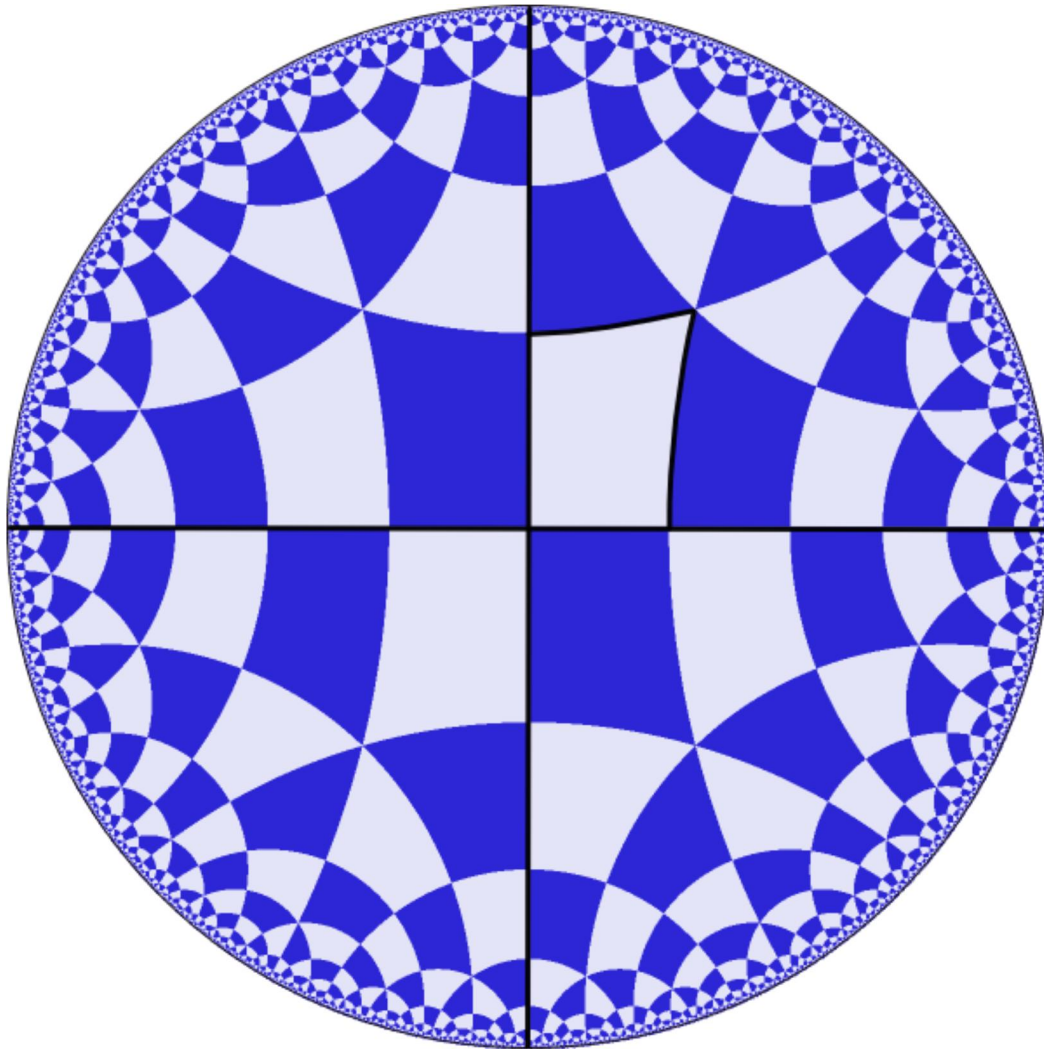
Quadrilateral kaleidocscope

13 reflections



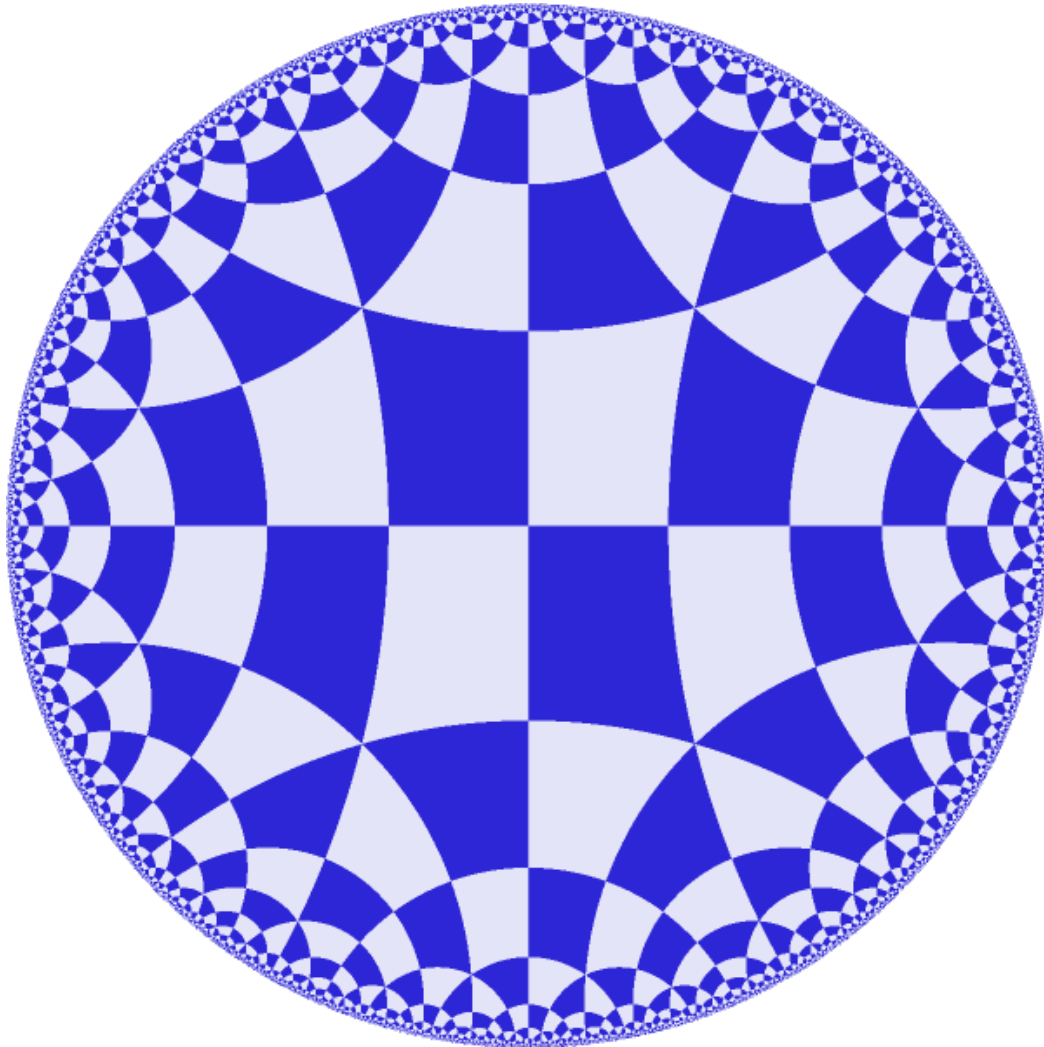
Quadrilateral kaleidocscope

15 reflections

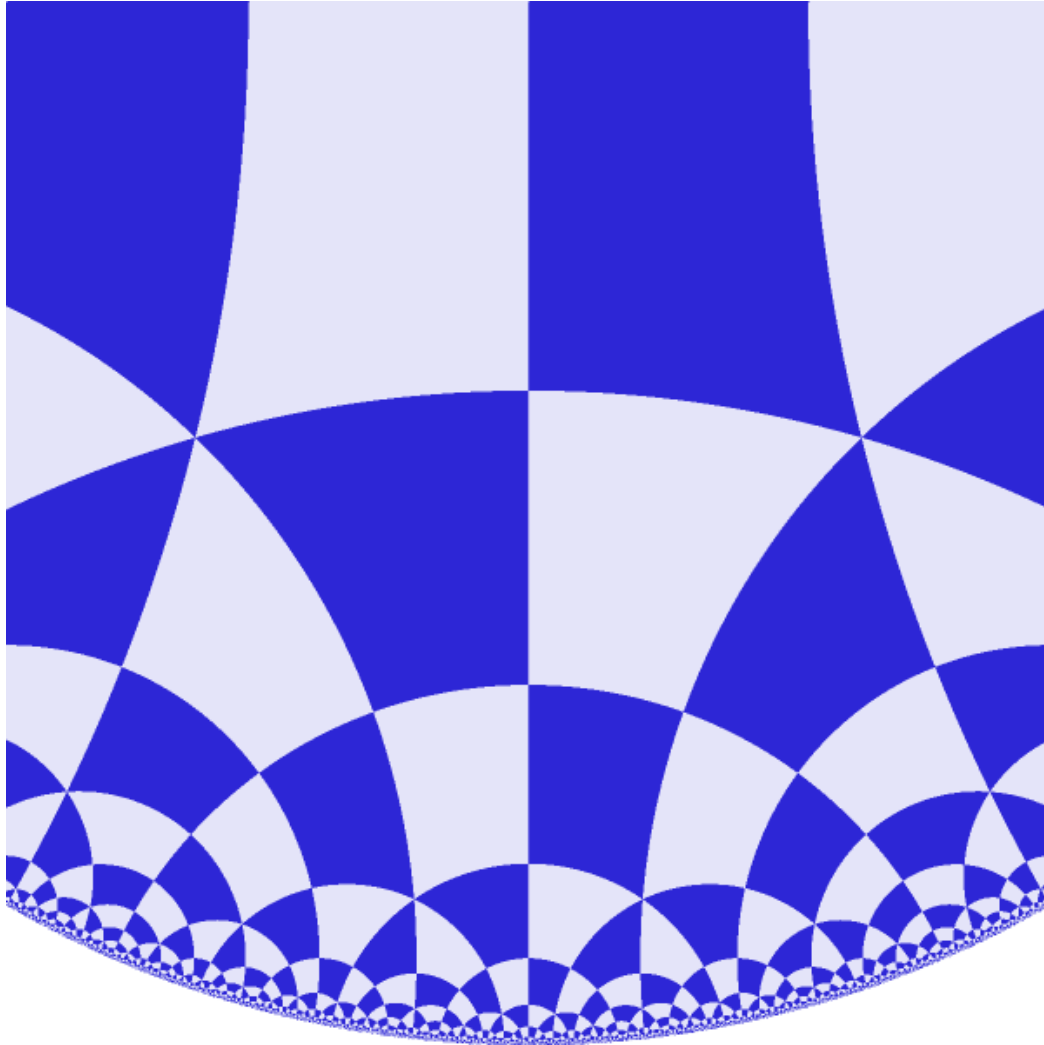


Quadrilateral kaleidocscope

Complete $*2223$ kaleidoscope

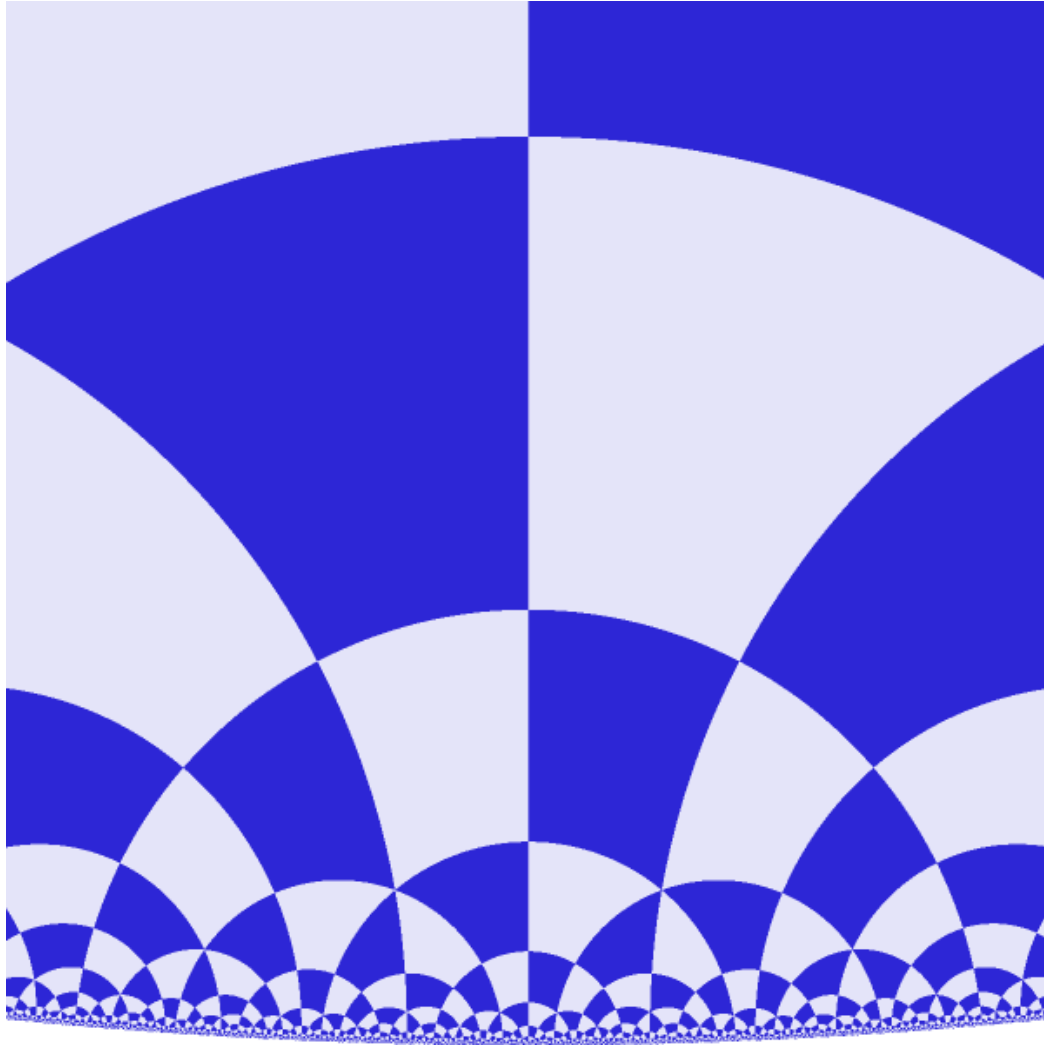


Quadrilateral kaleidocscope



Scale x 2.

Quadrilateral kaleidocscope



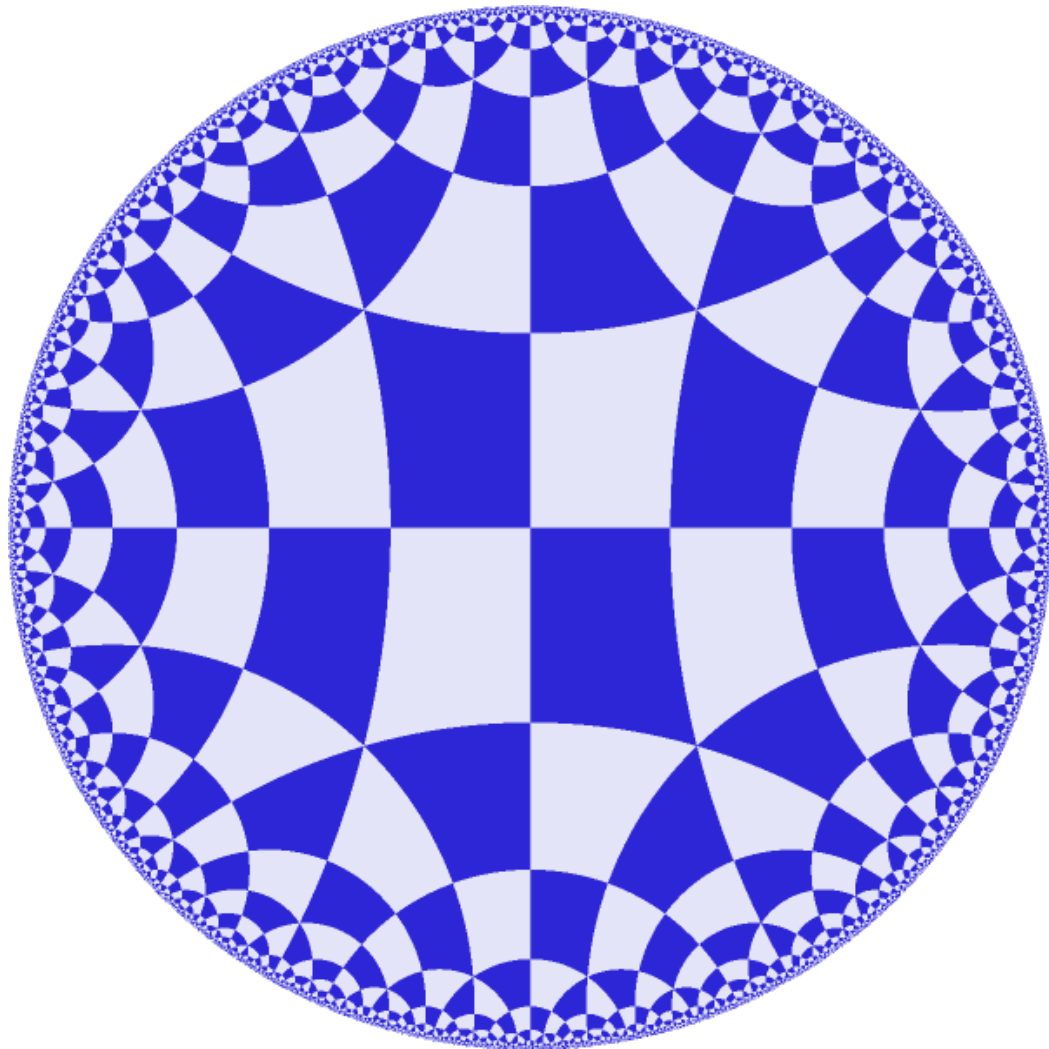
Scale x 10.

Quadrilateral kaleidocscope



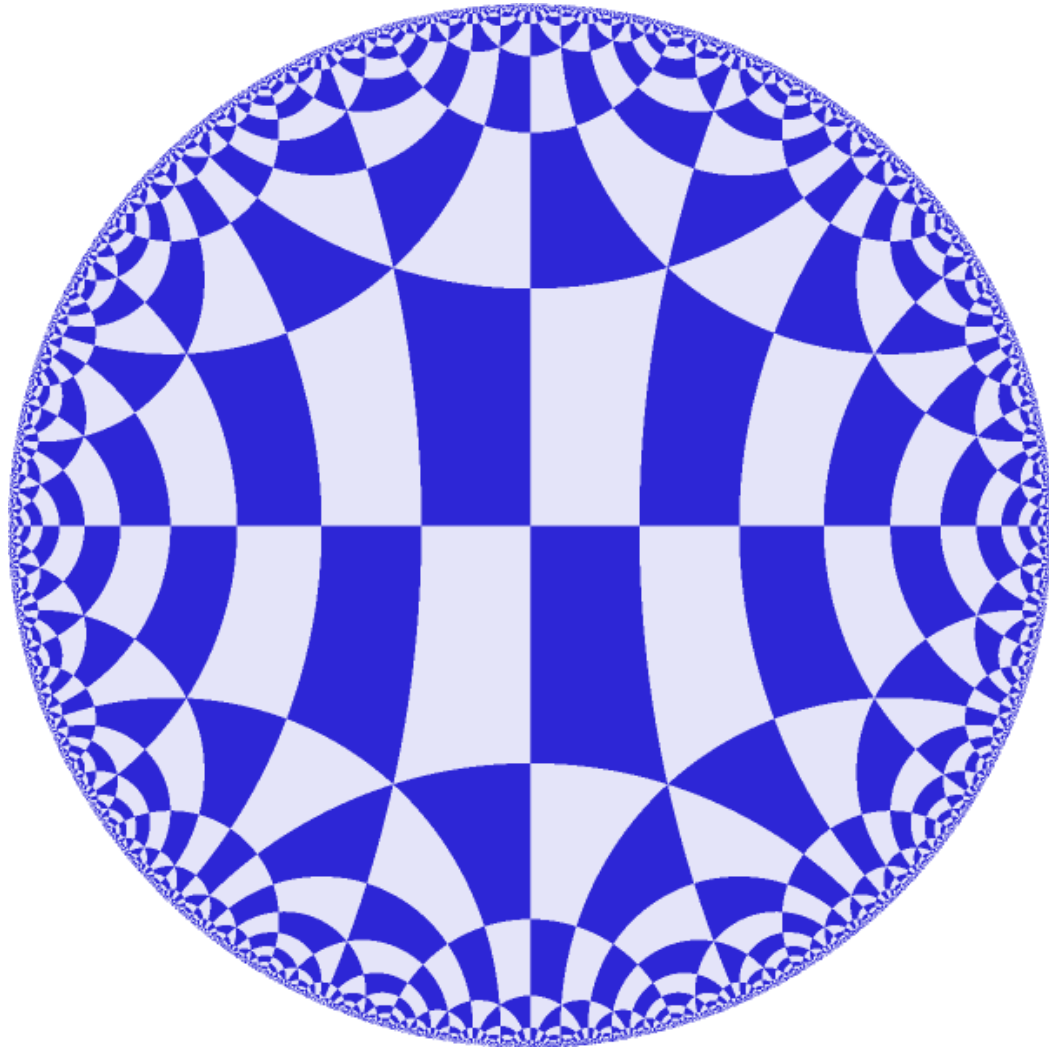
Scale x 50.

Stretching $*2223$ kaleidoscope



Length of one side of $*2223$ kaleidoscope is free parameter.

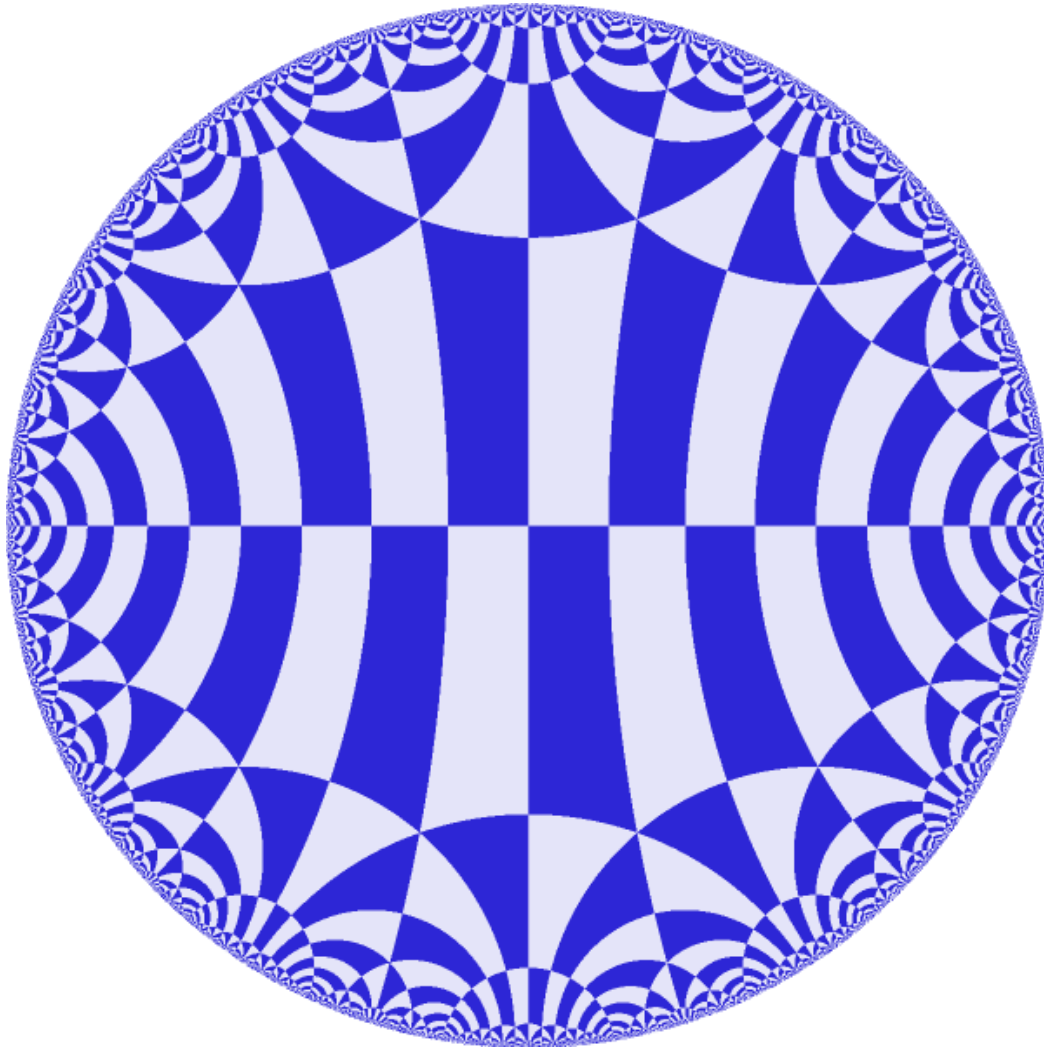
Stretching $*2223$ kaleidoscope



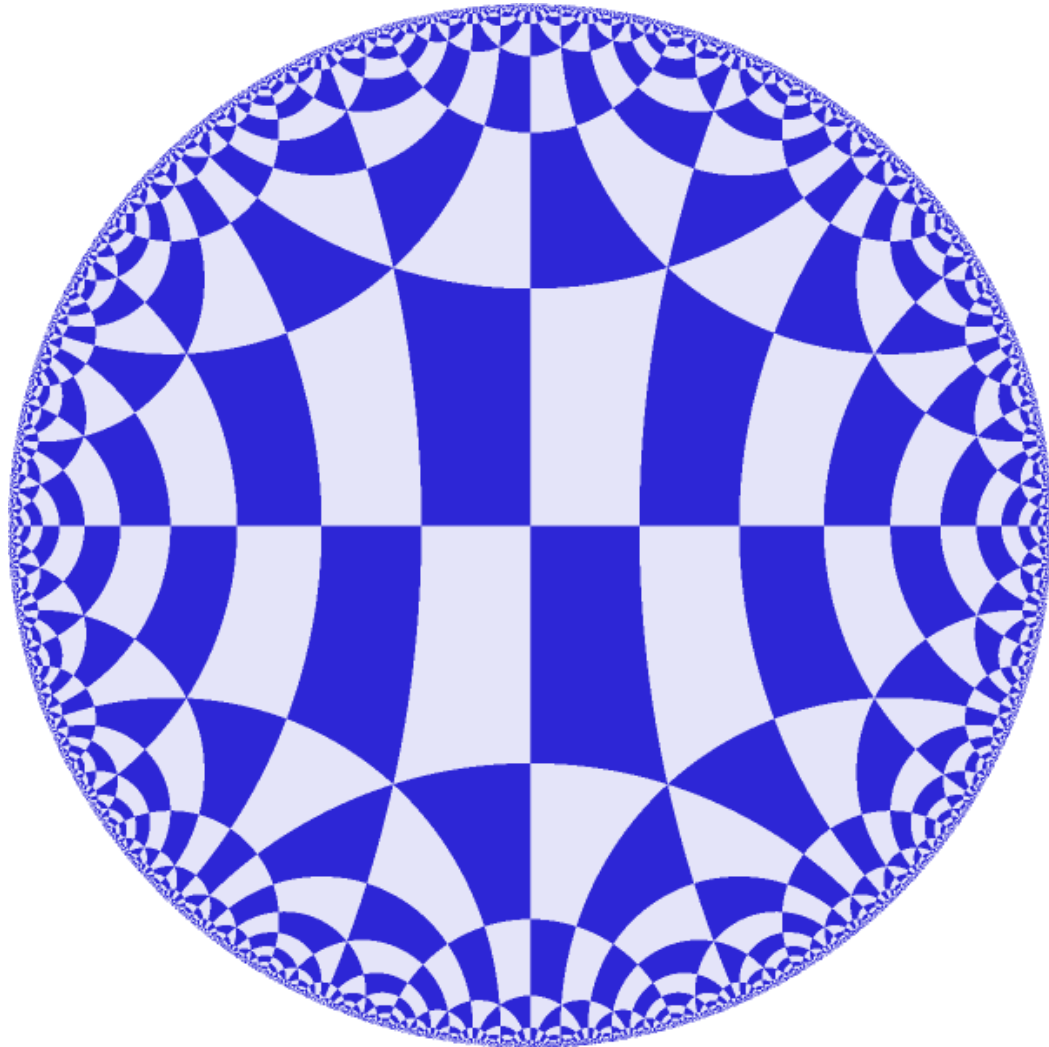
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope

Length of one side of $*2223$ kaleidoscope
is free parameter.

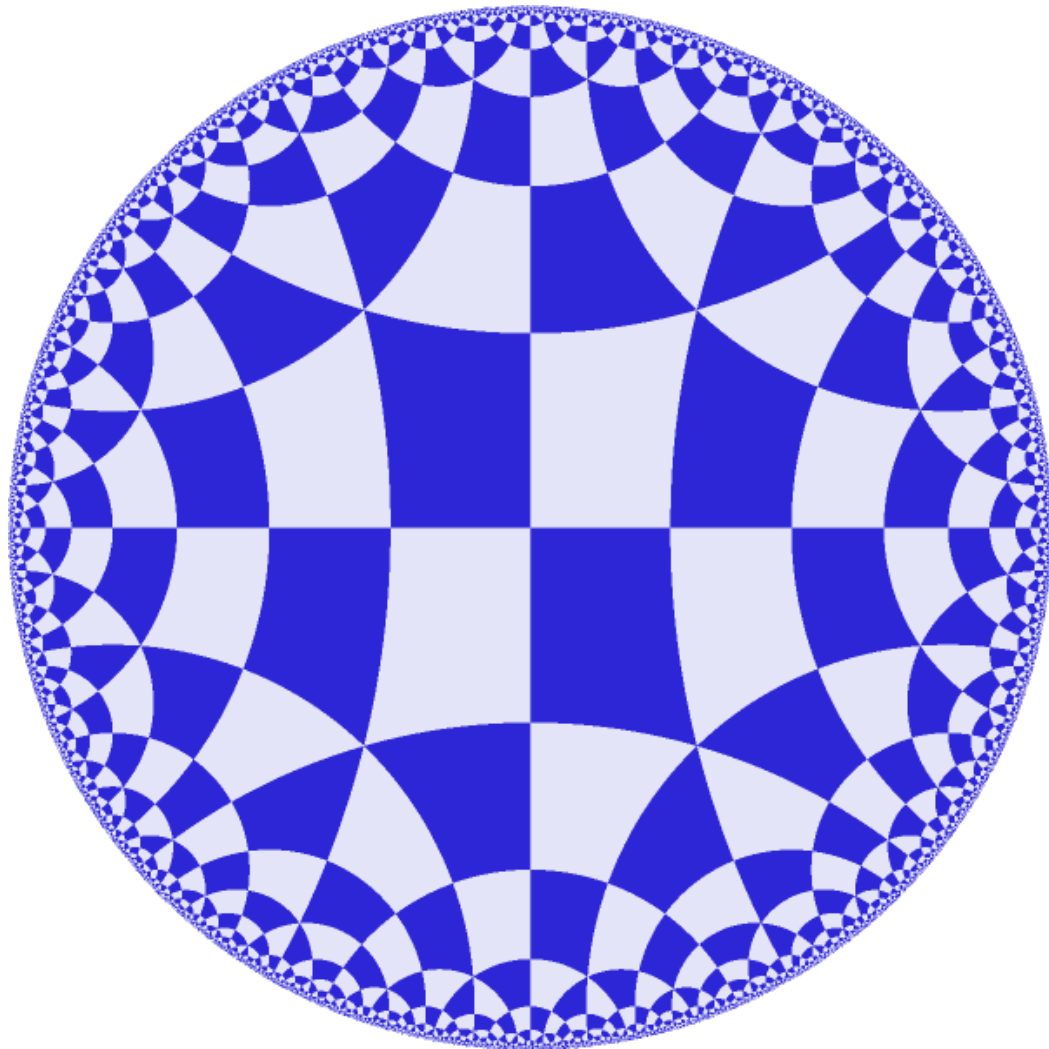


Stretching $*2223$ kaleidoscope



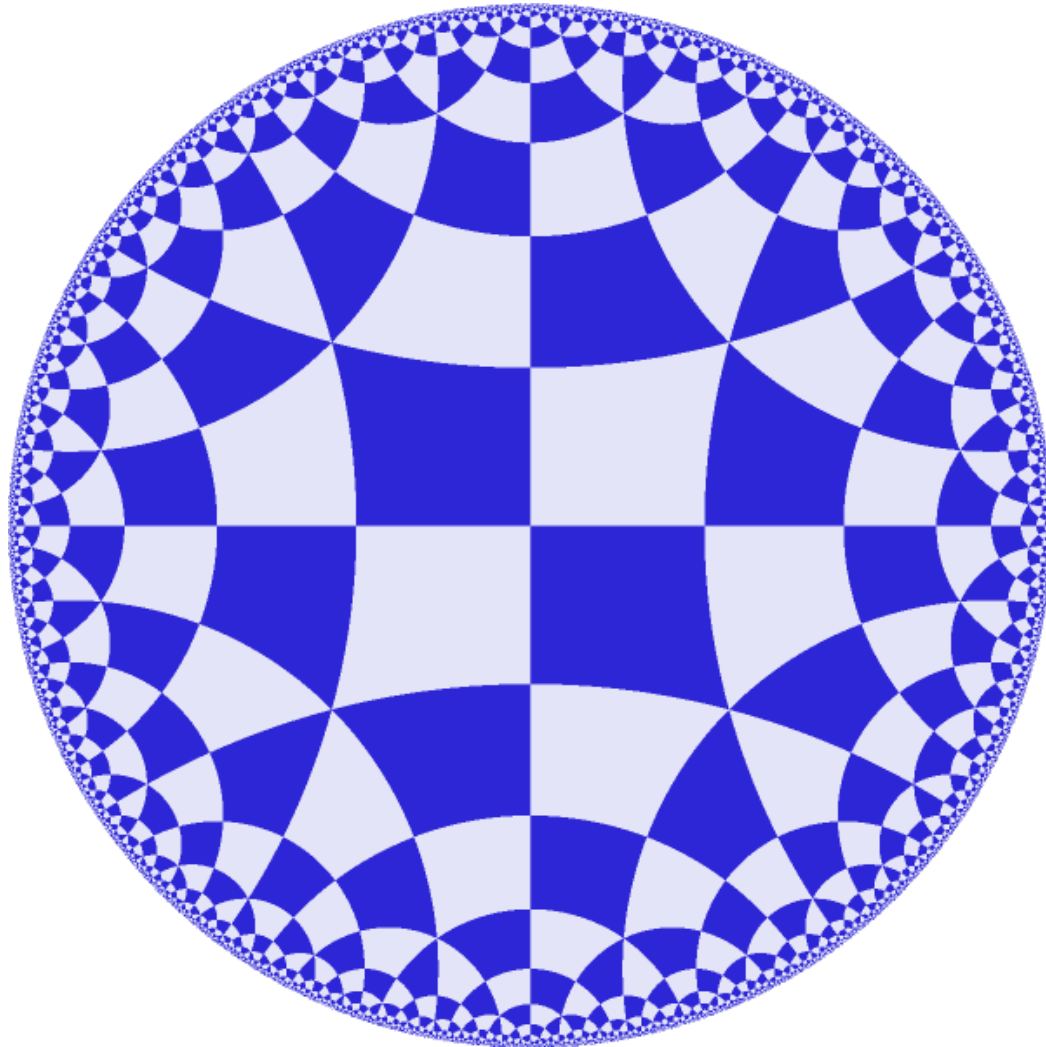
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope



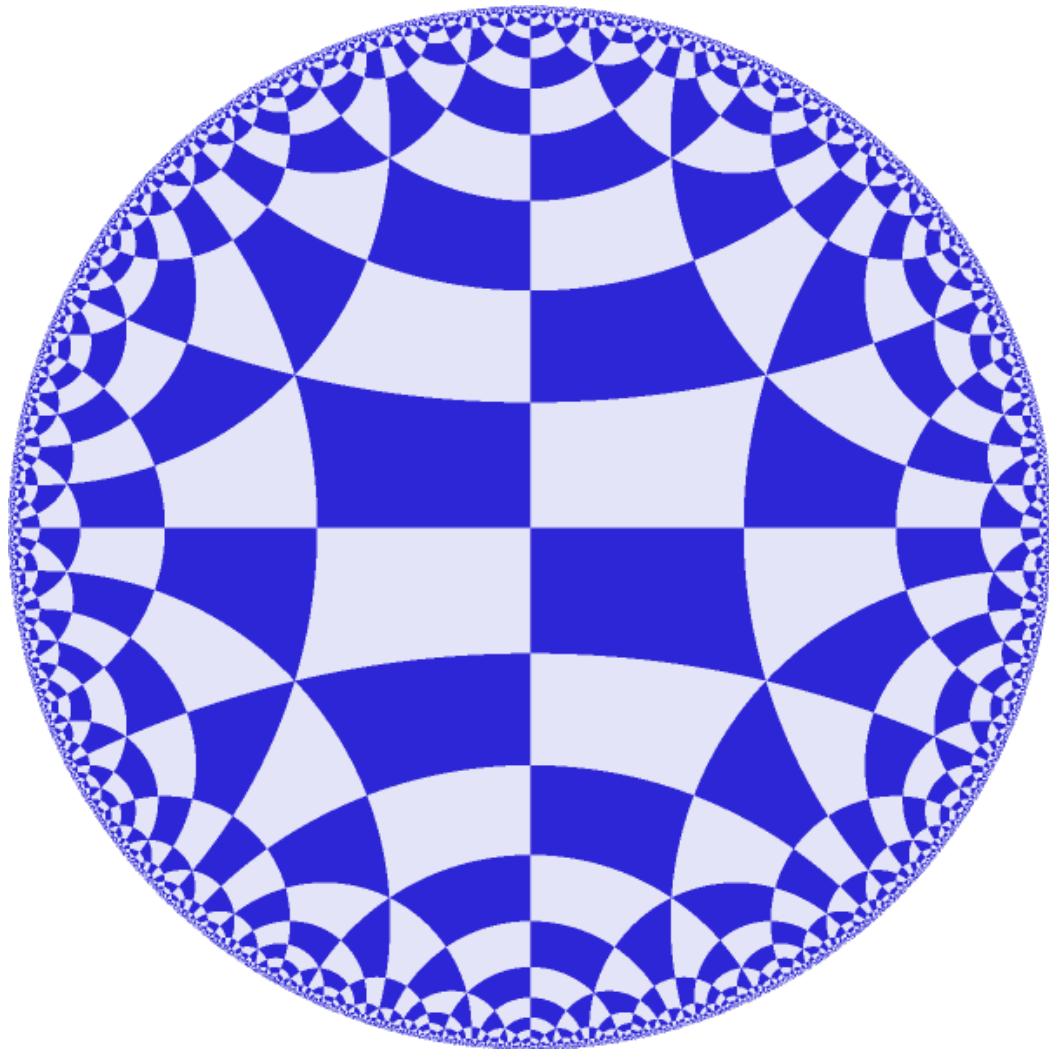
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope



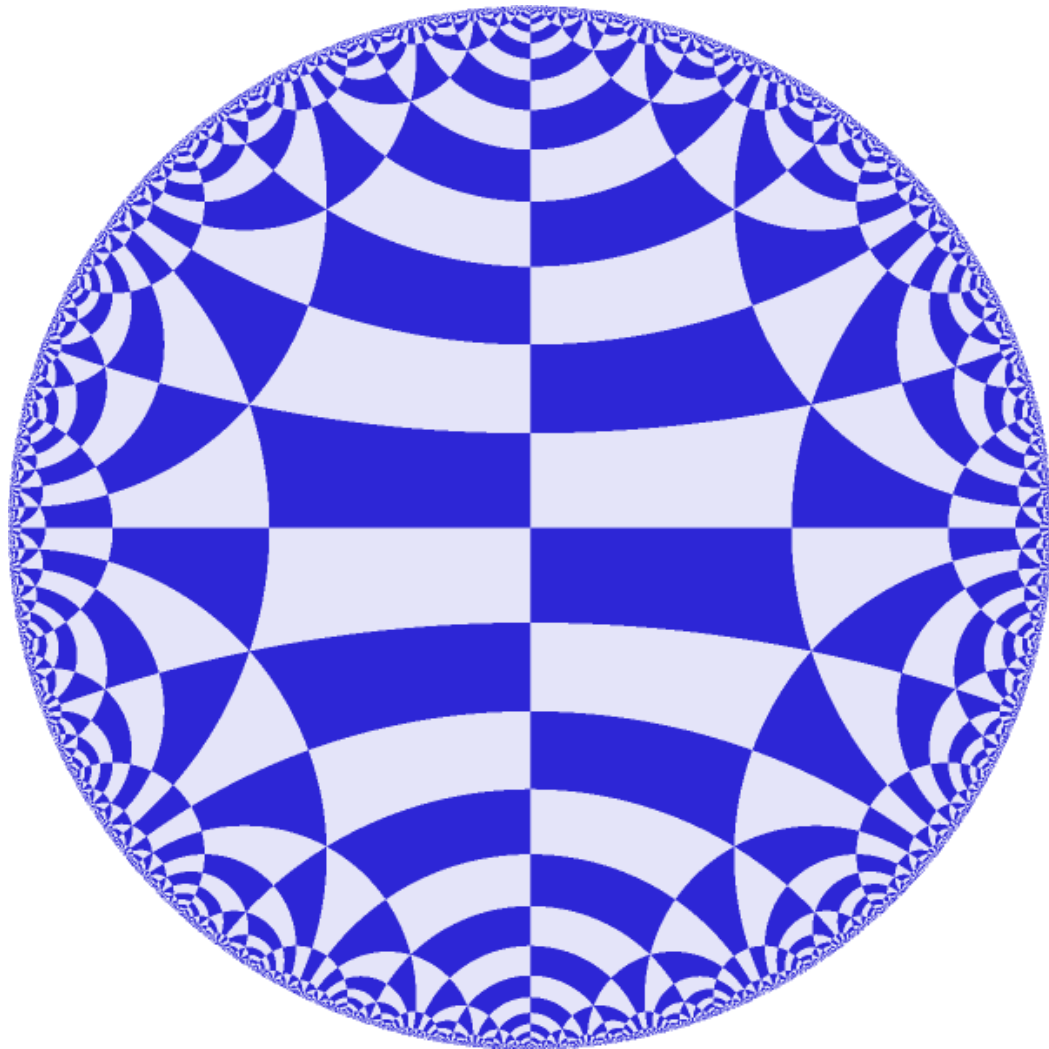
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope



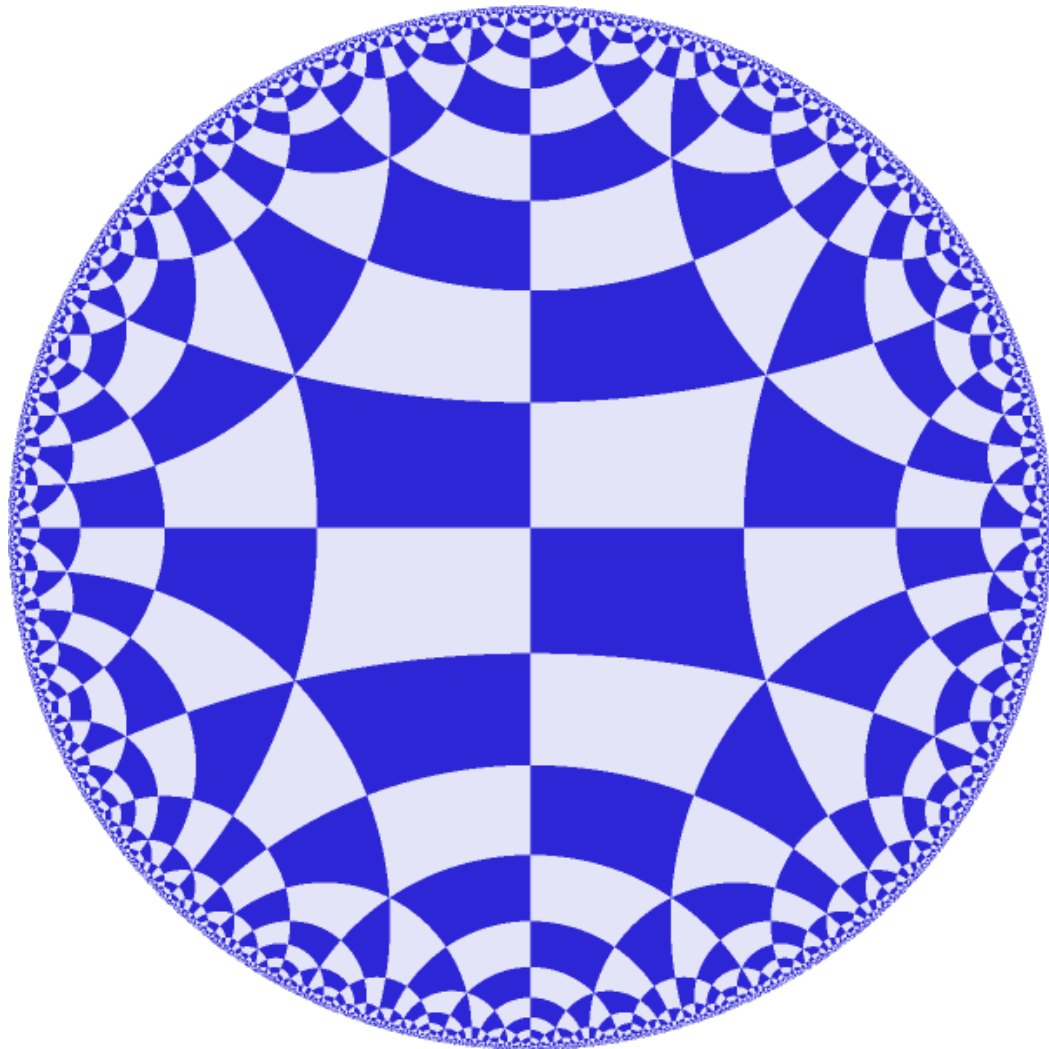
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope



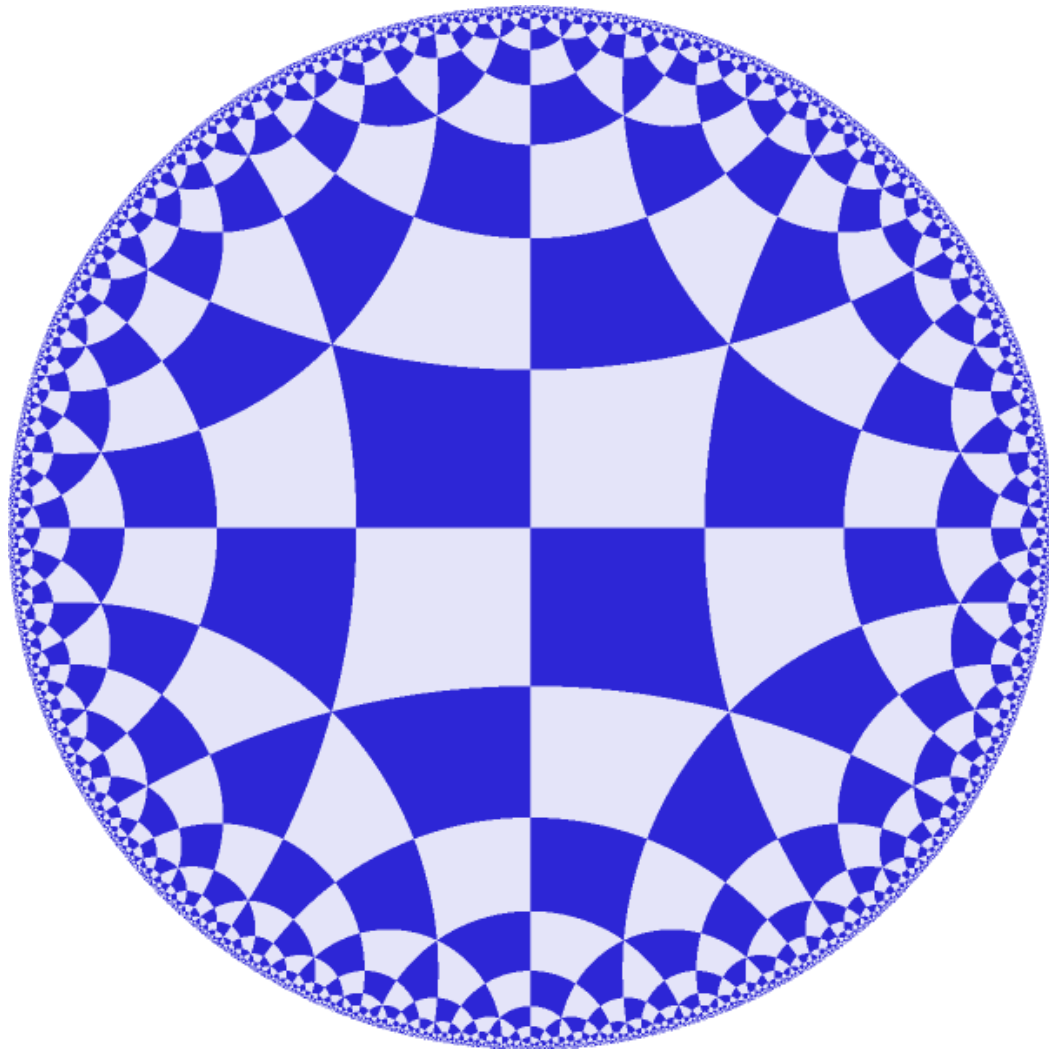
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope



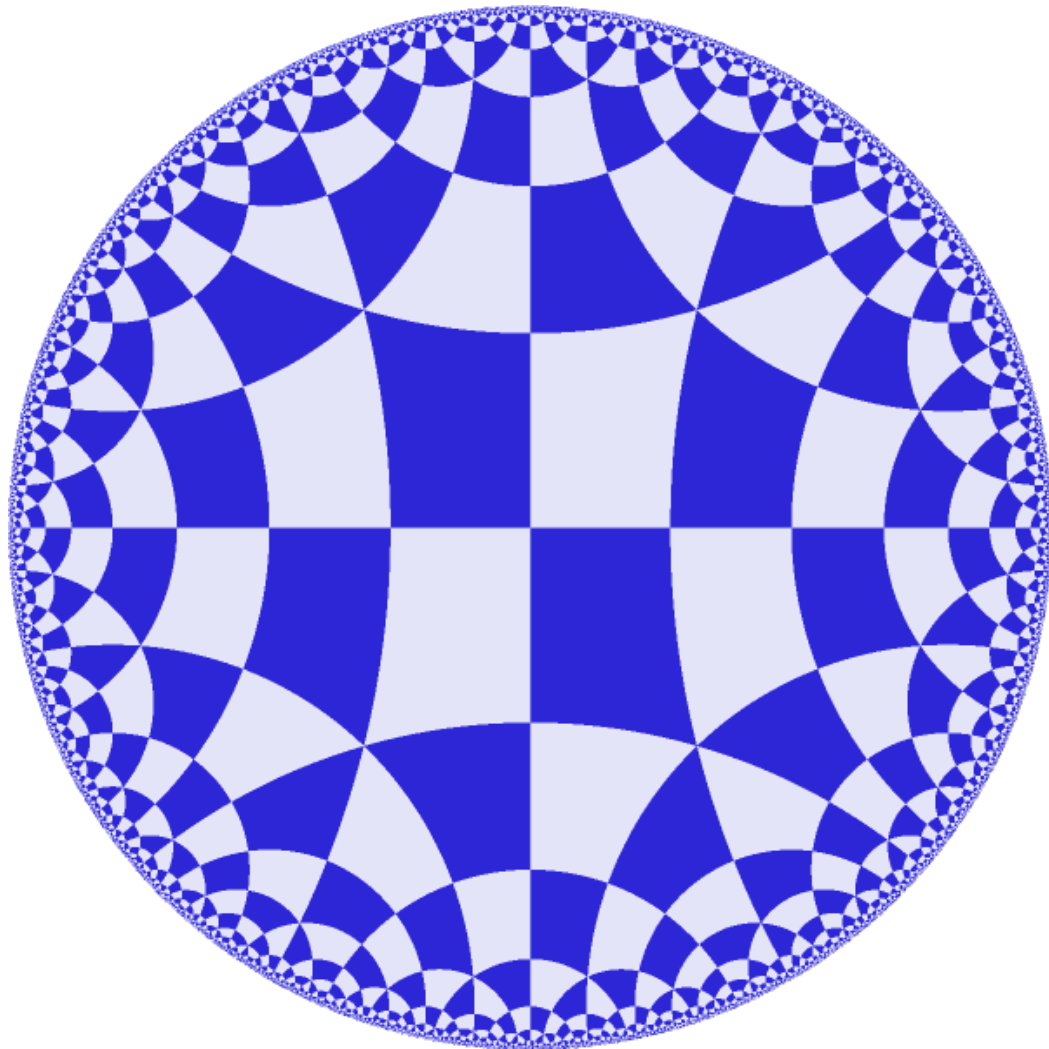
Length of one side of $*2223$ kaleidoscope
is free parameter.

Stretching $*2223$ kaleidoscope



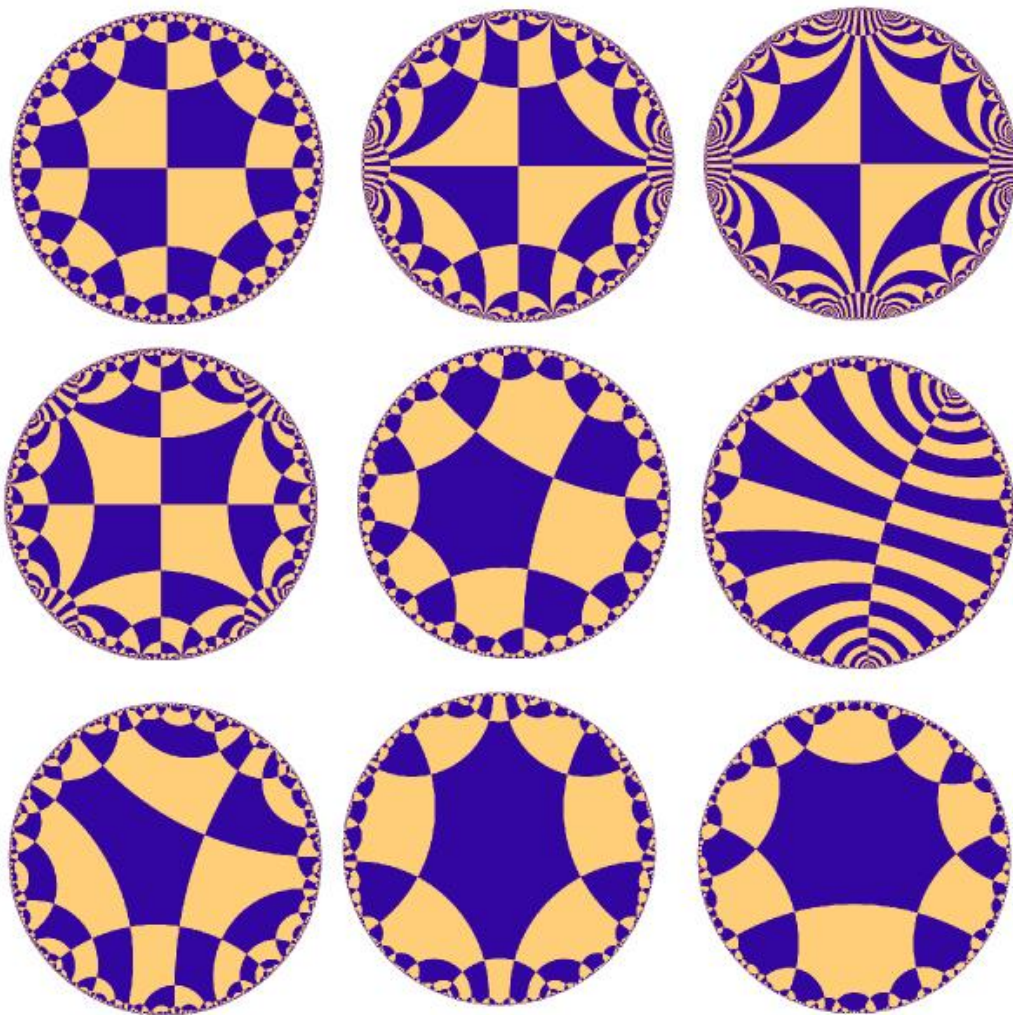
Length of one side of $*2223$ kaleidoscope is free parameter.

Stretching $*2223$ kaleidoscope



Length of one side of $*2223$ kaleidoscope is free parameter.

Arbitrary n-gonal *NML... kaleidoscopes

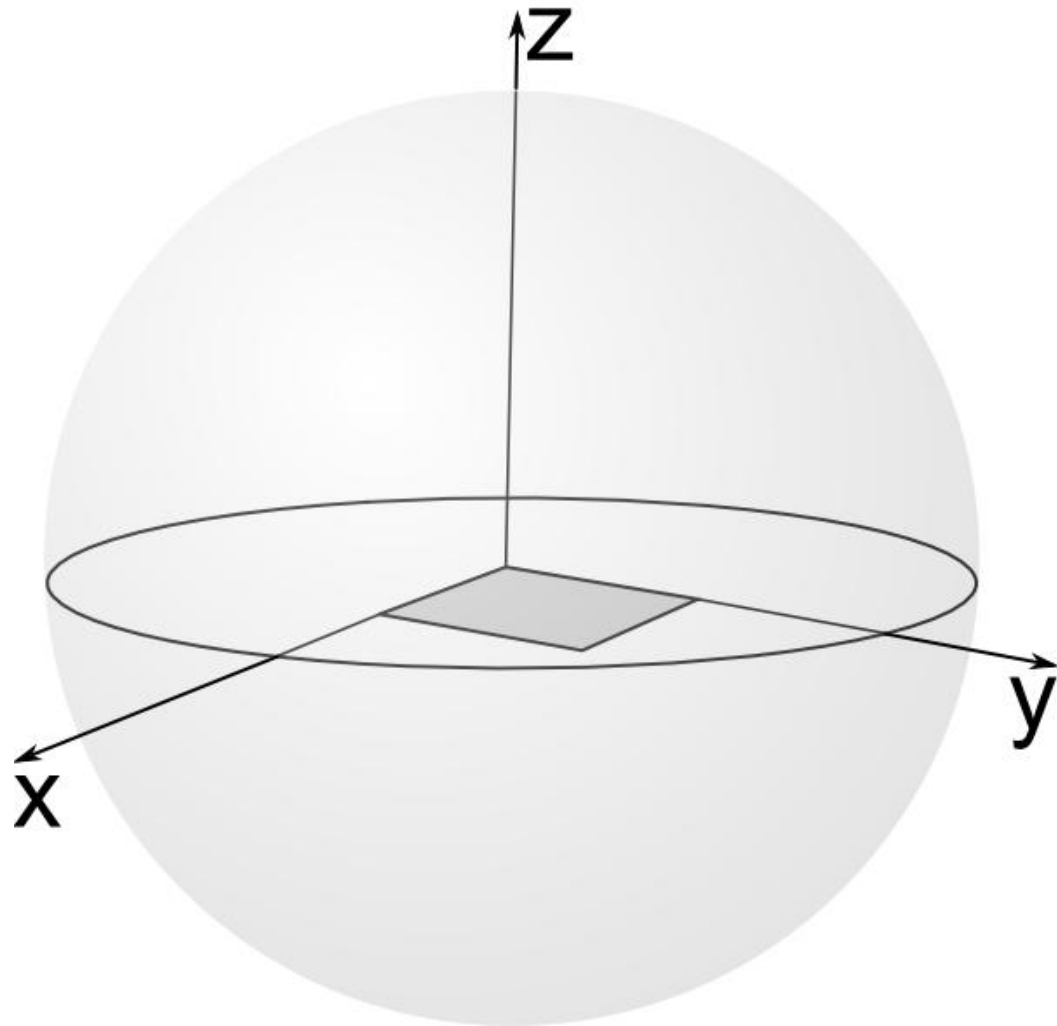


Any hyperbolic n -gon with "kaleidoscopic" angles generates kaleidoscope.

Such kaleidoscope has $n - 3$ free continuous parameters (lengths of $n - 3$ sides for example).

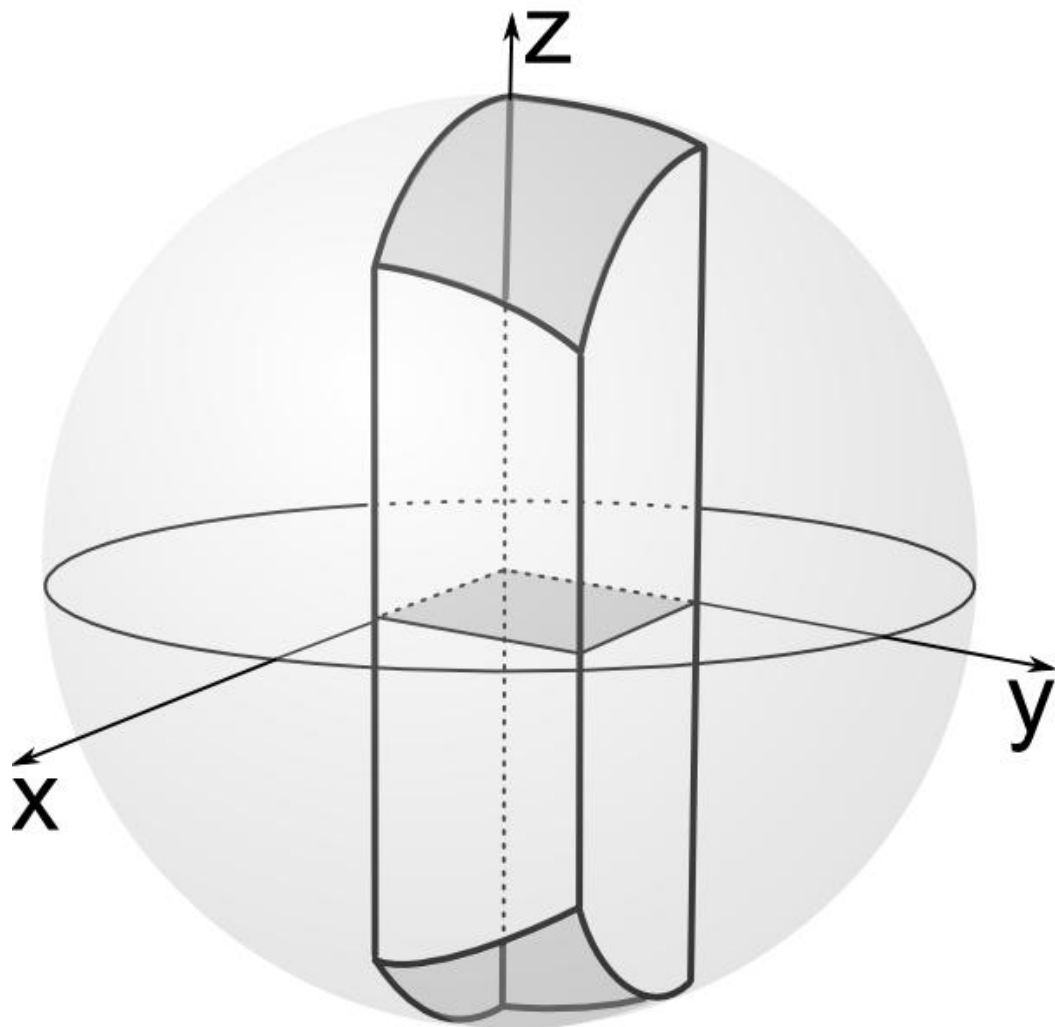
Not for all values of parameters such n -gon exists.

Go to 3 dimensions



Mapping unit disk to unit sphere. Klein model.

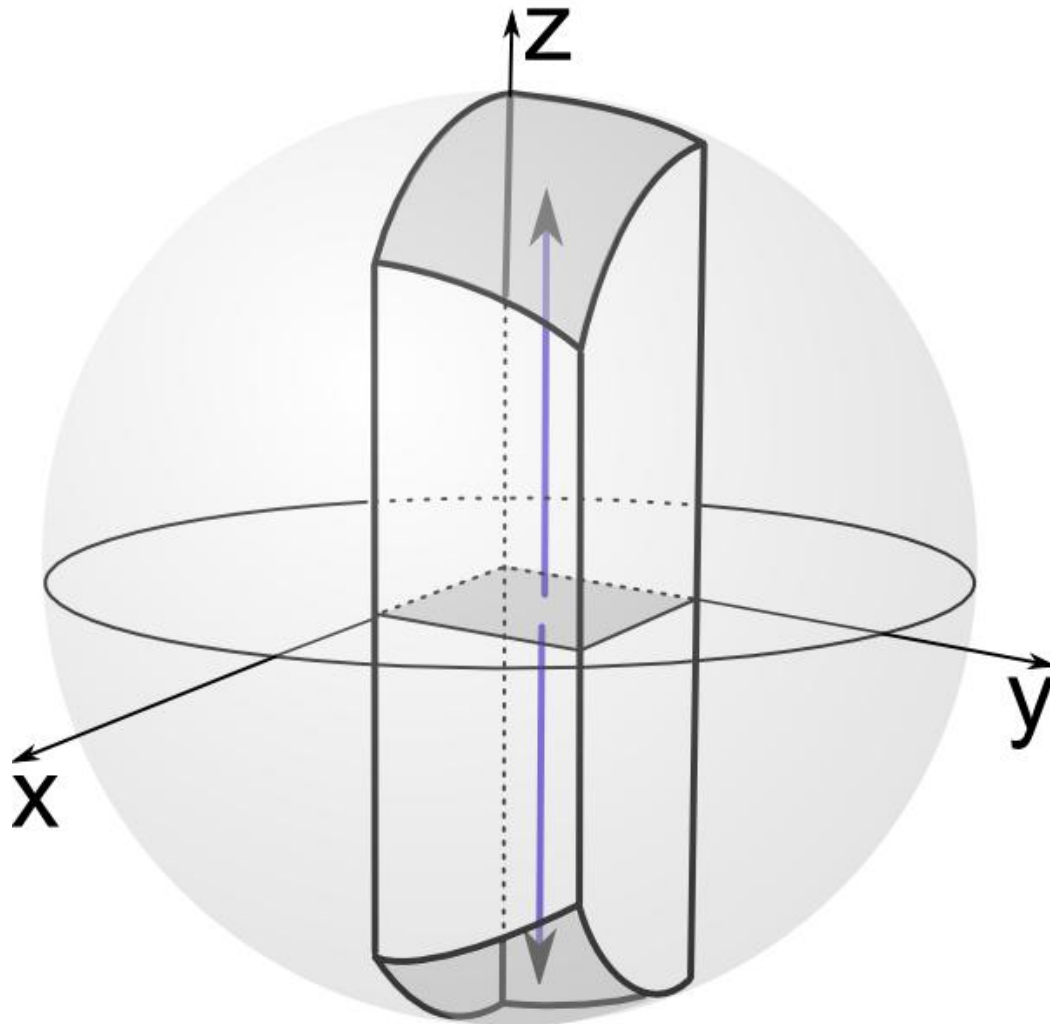
Go to 3 dimensions



Mapping unit disk to unit sphere. Klein model.

Extend flat quadrilateral in x - y plane into 3-dimensional chimney orthogonal to x - y plane.

Go to 3 dimensions



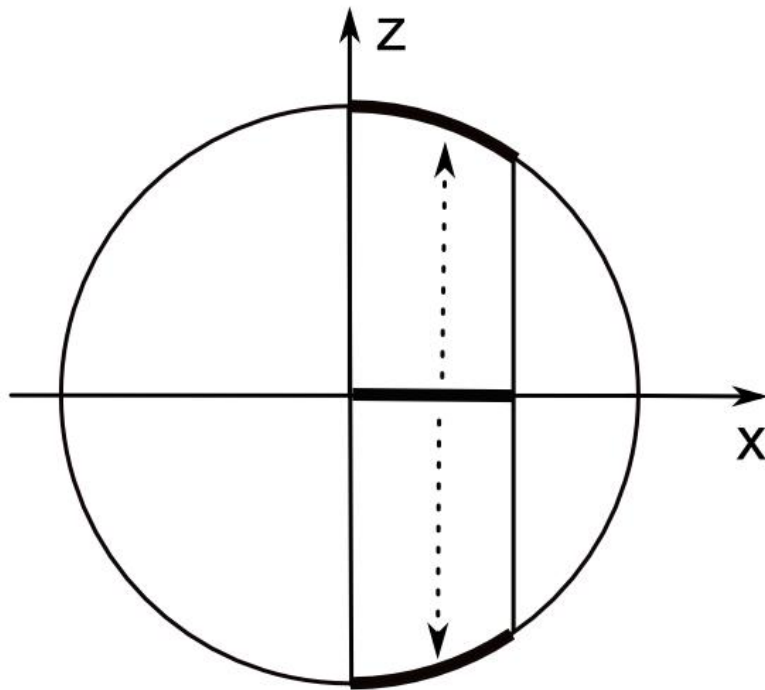
Mapping unit disk to unit sphere. Klein model.

Extend flat quadrilateral in x - y plane into 3-dimensional chimney orthogonal to x - y plane.

Map quadrilateral into 2 spherical quadrilaterals.

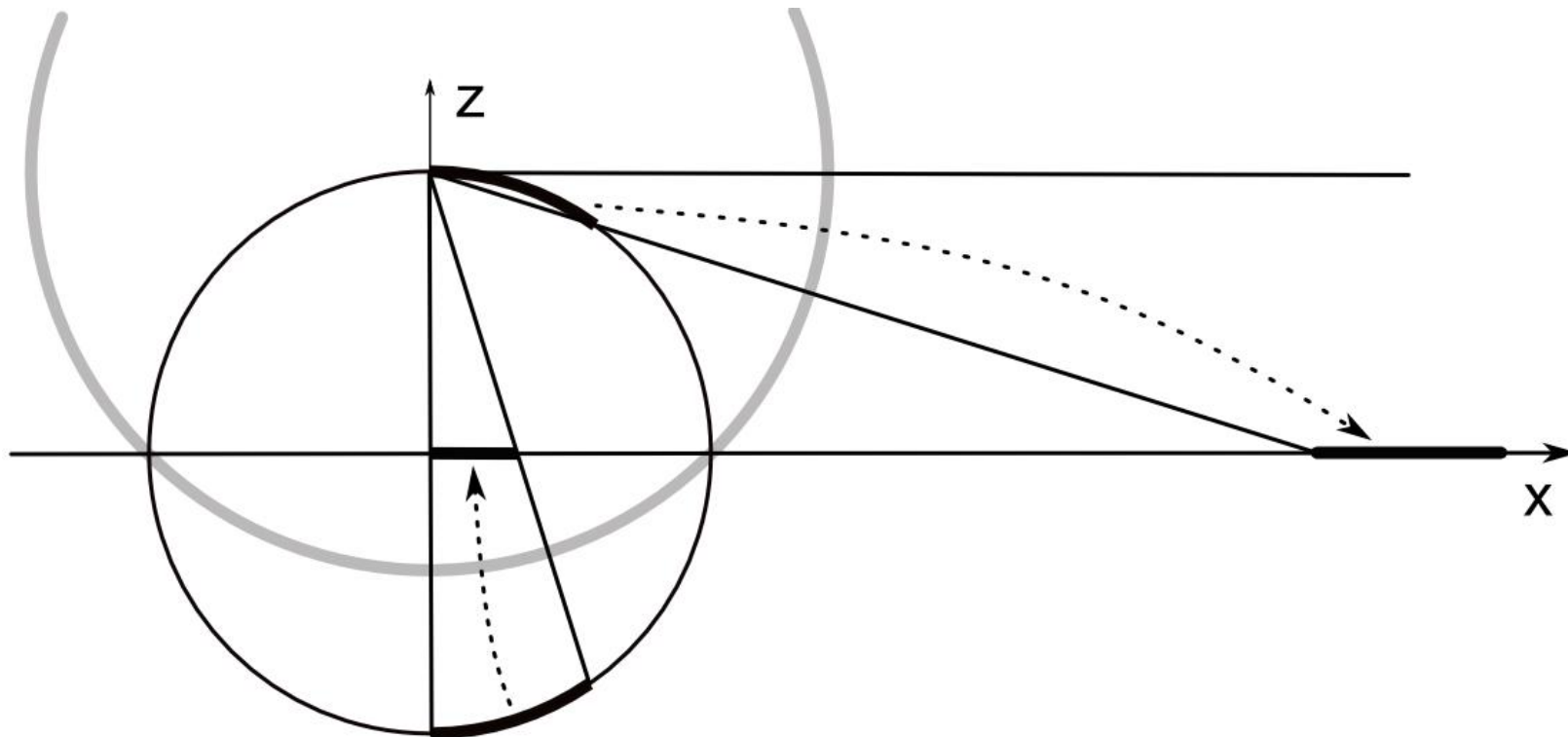
Go to 3 dimensions

Mapping unit disk to unit sphere, y-axis view.

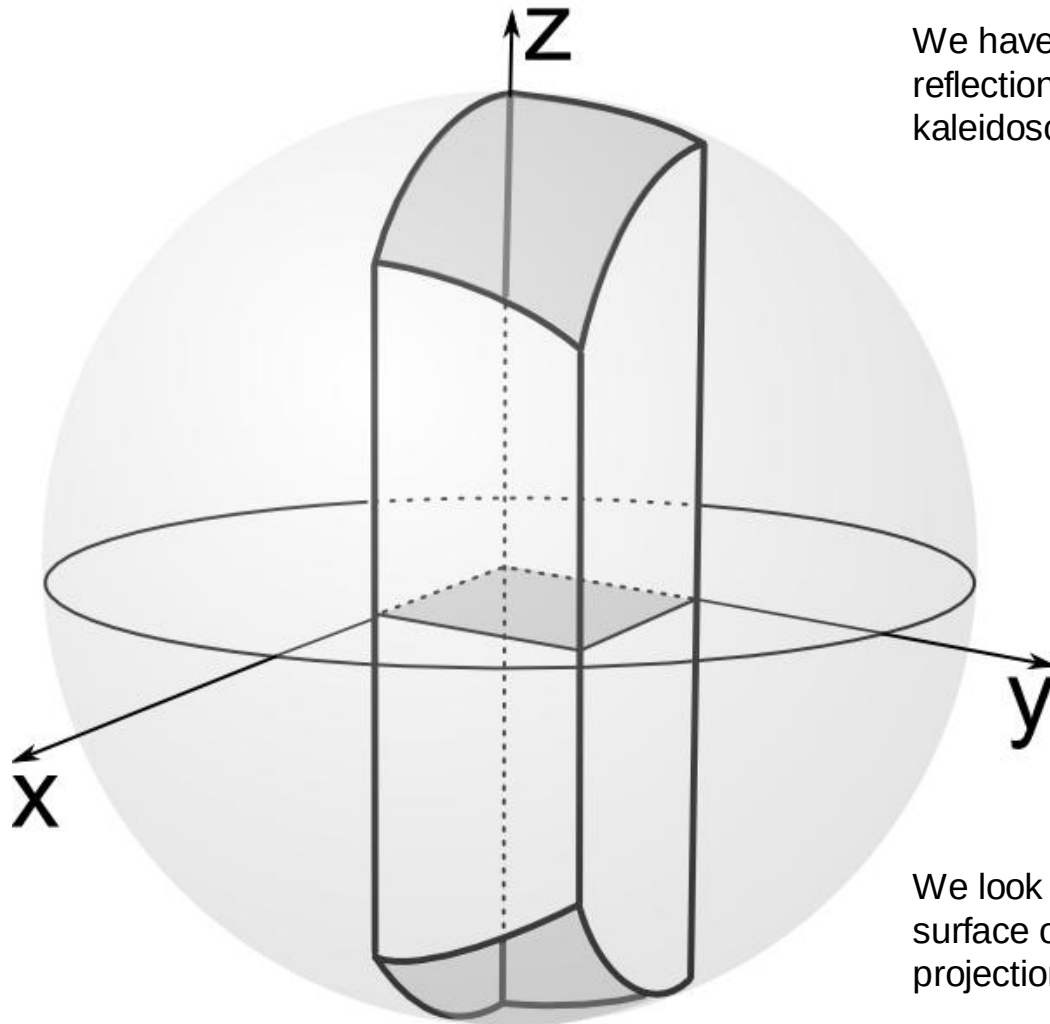


Go to 3 dimensions

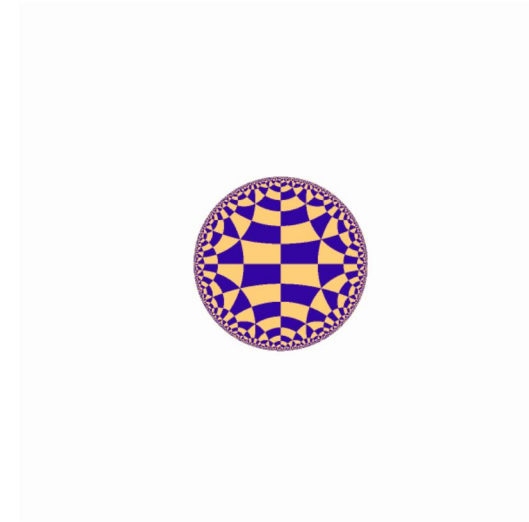
Stereographic projection back into plane.



*2223 kaleidoscope

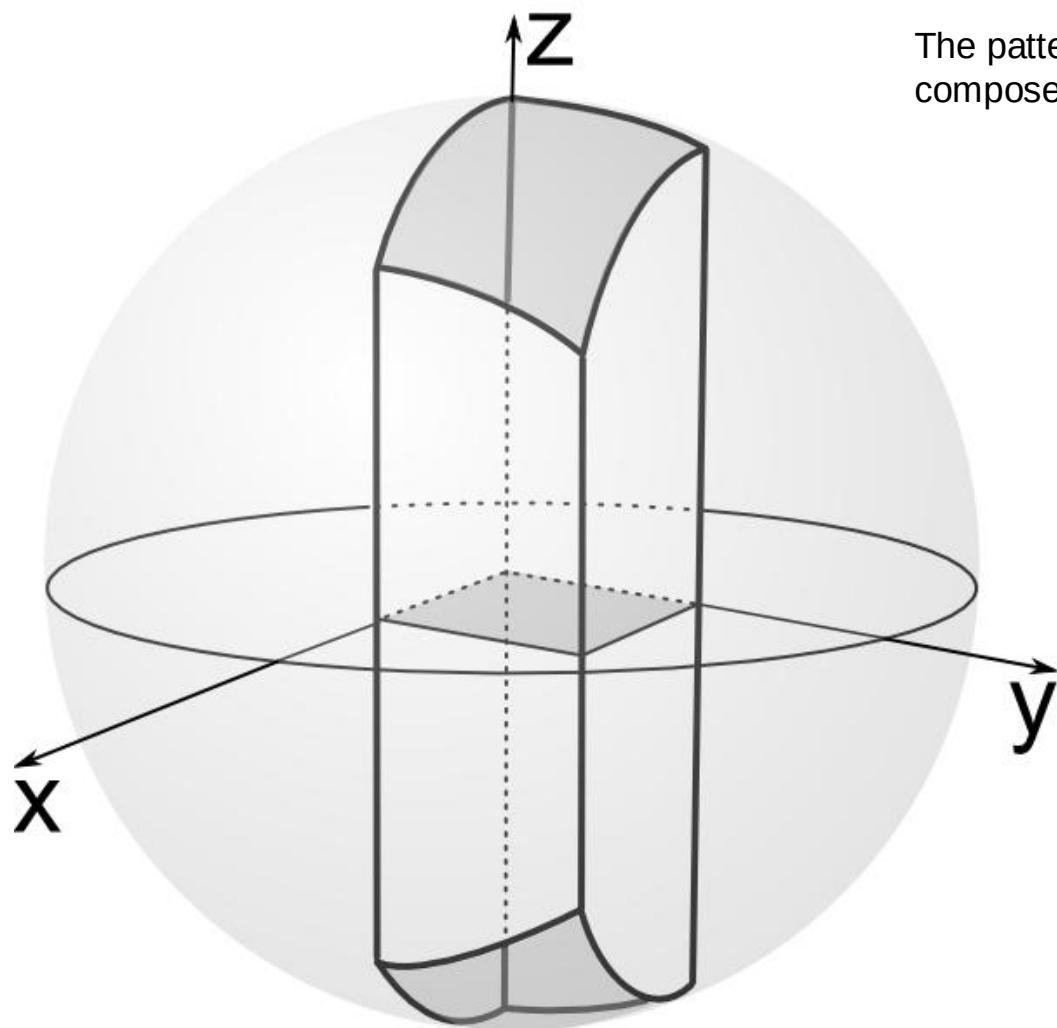


We have quadrilateral hyperbolic chimney. Hyperbolic reflections in it's sides generate 3-dimensional hyperbolic kaleidoscope.

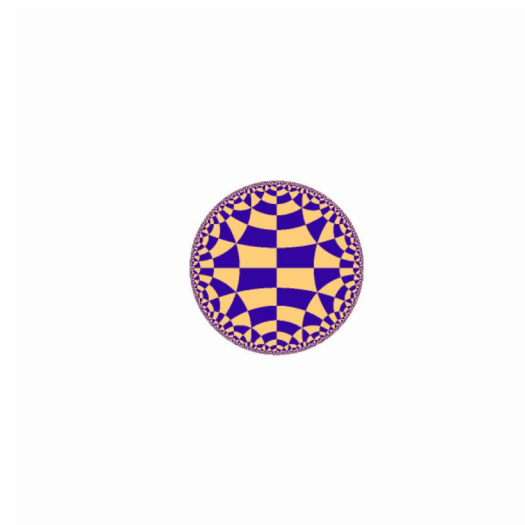


We look at the pattern created by this kaleidoscope on the surface of Poincare ball mapped to a plane via stereographic projection.

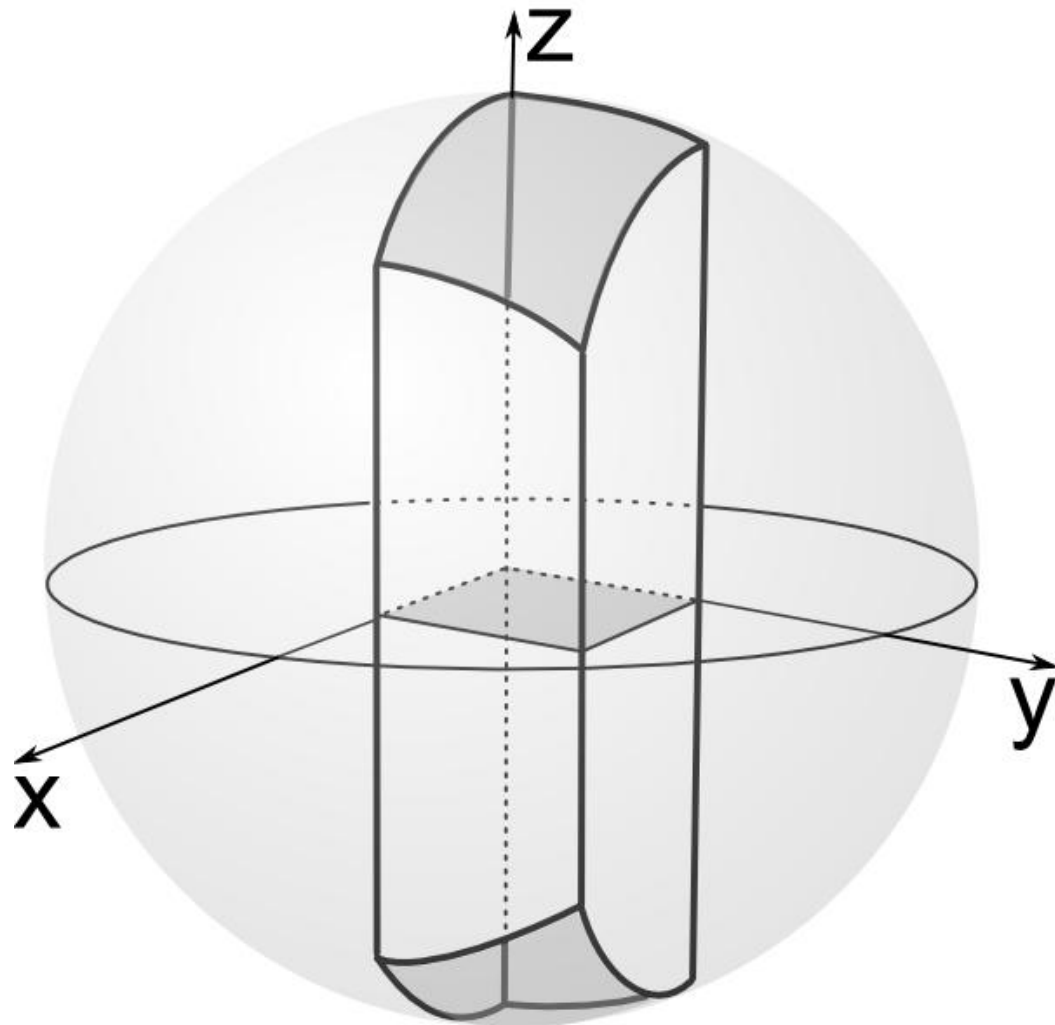
***2223 kaleidoscope**



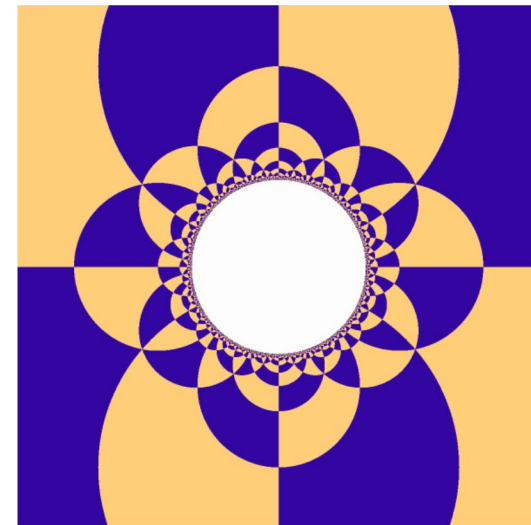
The pattern has two component: inside of a circle is composed of images of the bottom of the chimney.



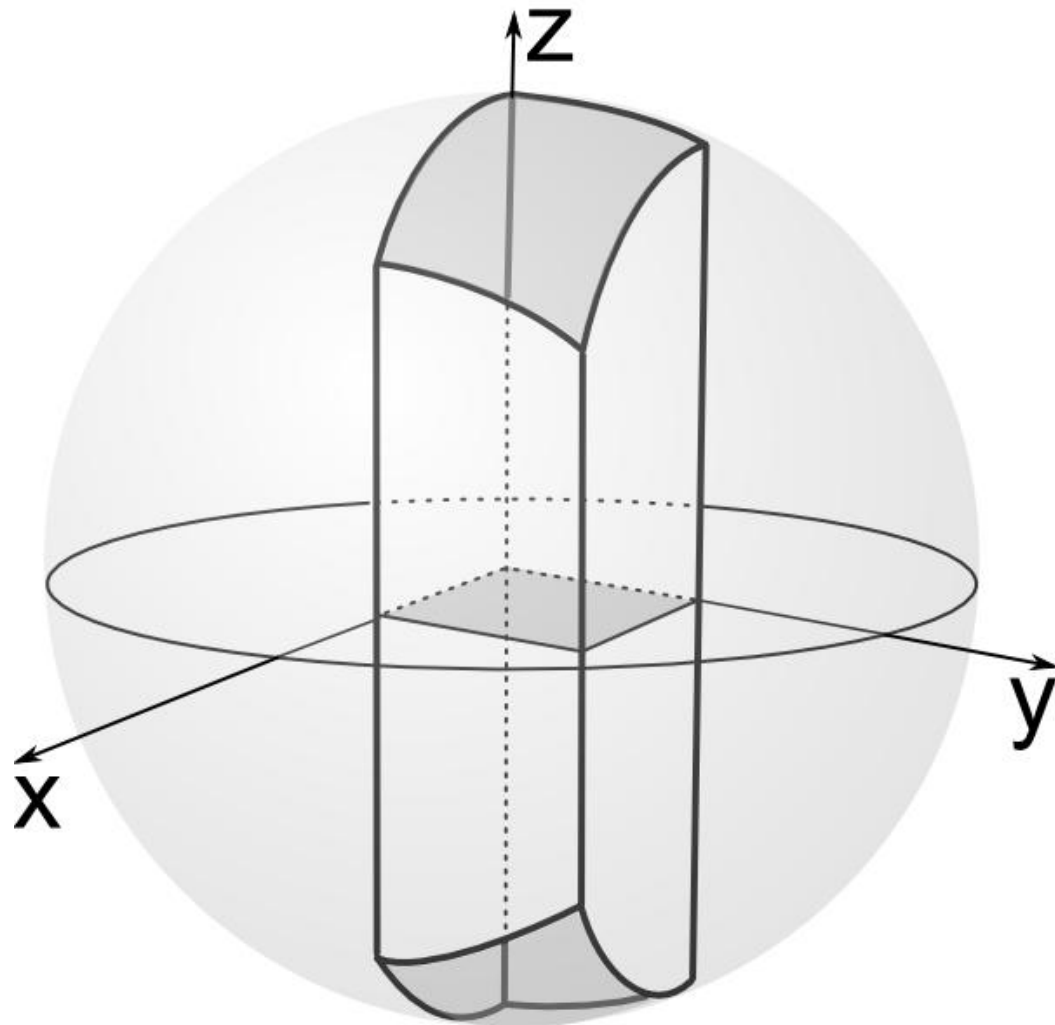
*2223 kaleidoscope



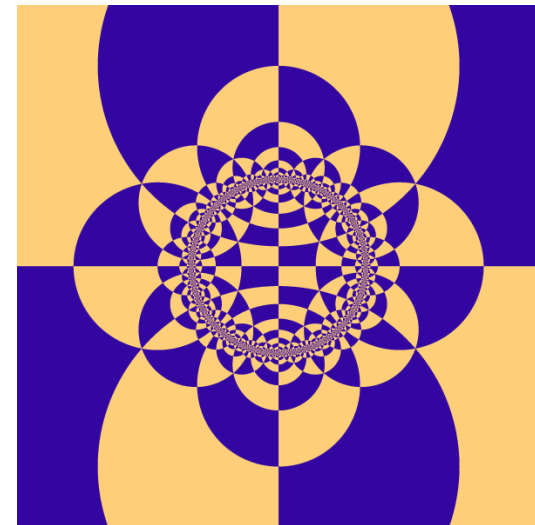
Outside of the the circle is composed from images of the top of the chimney.



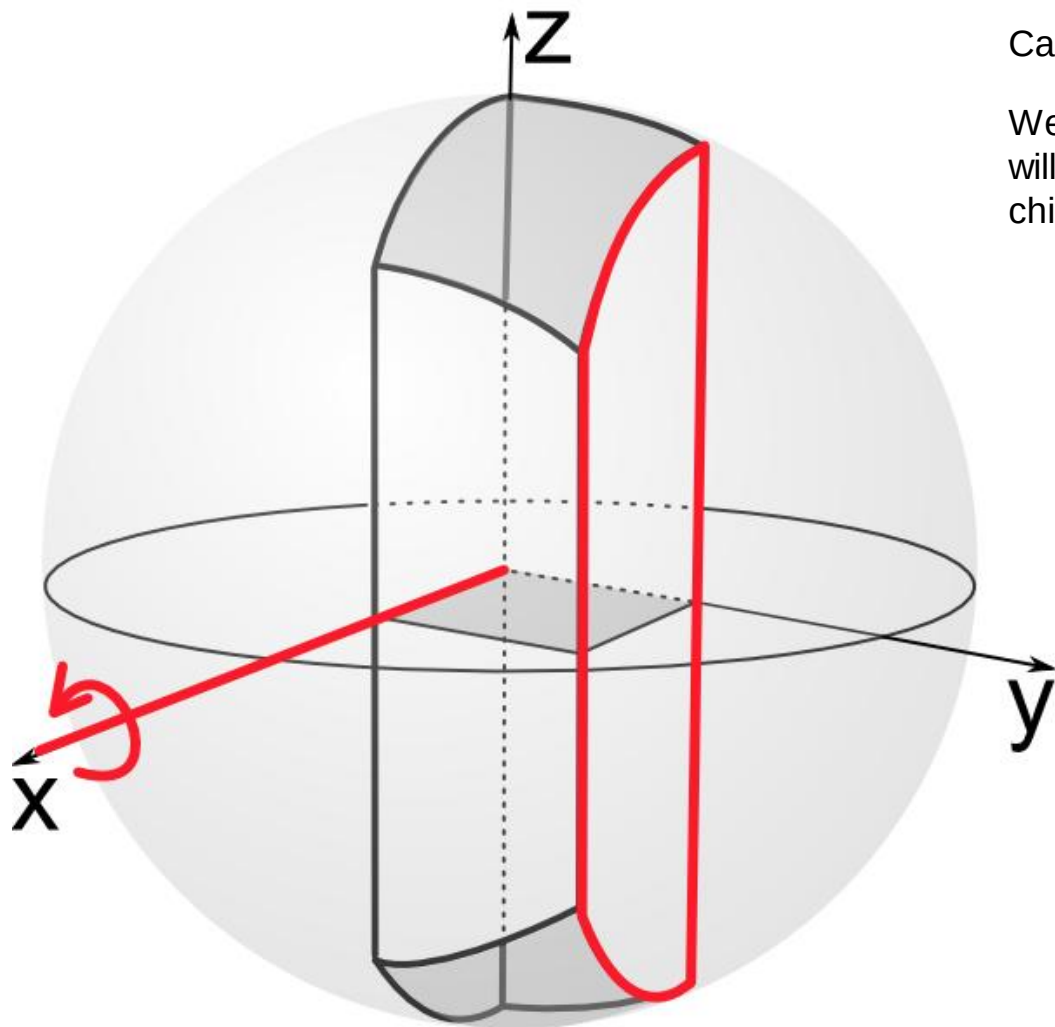
***2223 kaleidoscope**



Combined kaleidoscope image.

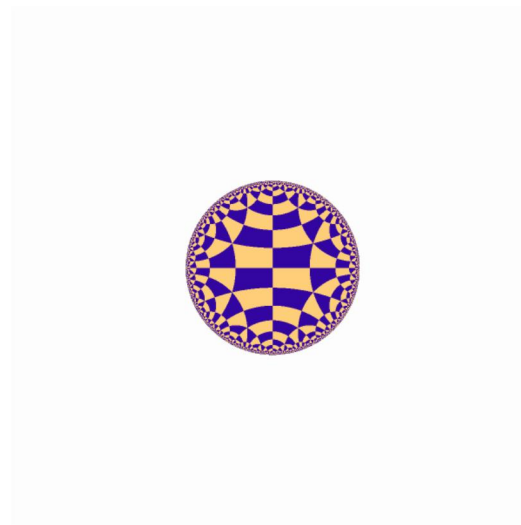


Bending *2223 kaleidoscope

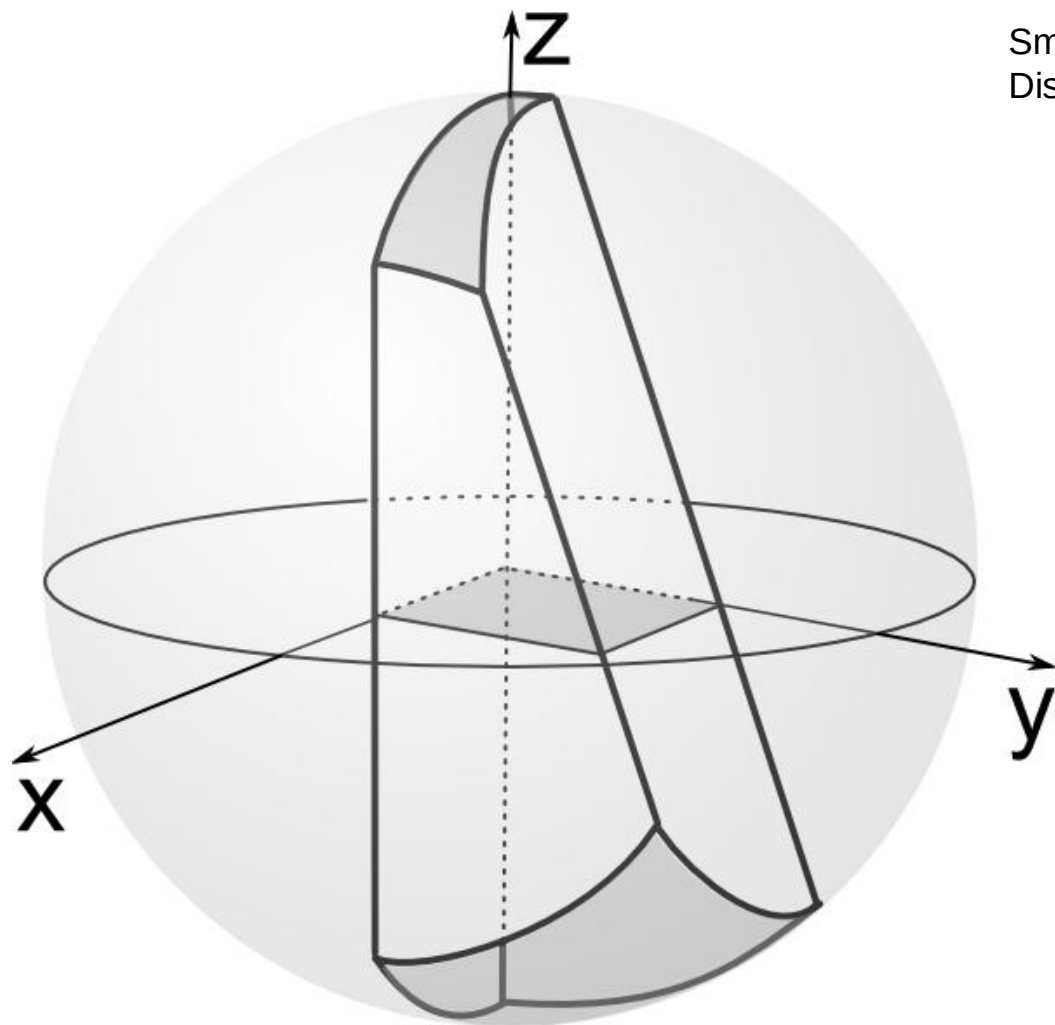


Can we deform *2223 kaleidoscope in 3 dimensions?

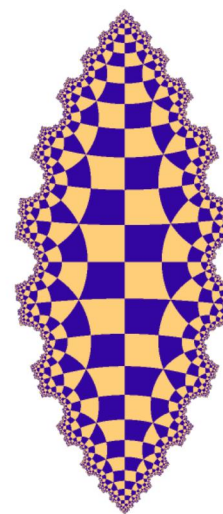
We can rotate selected plane about x-axis. This motion will not change dihedral angles of this side of the chimney with its neighbouring sides.



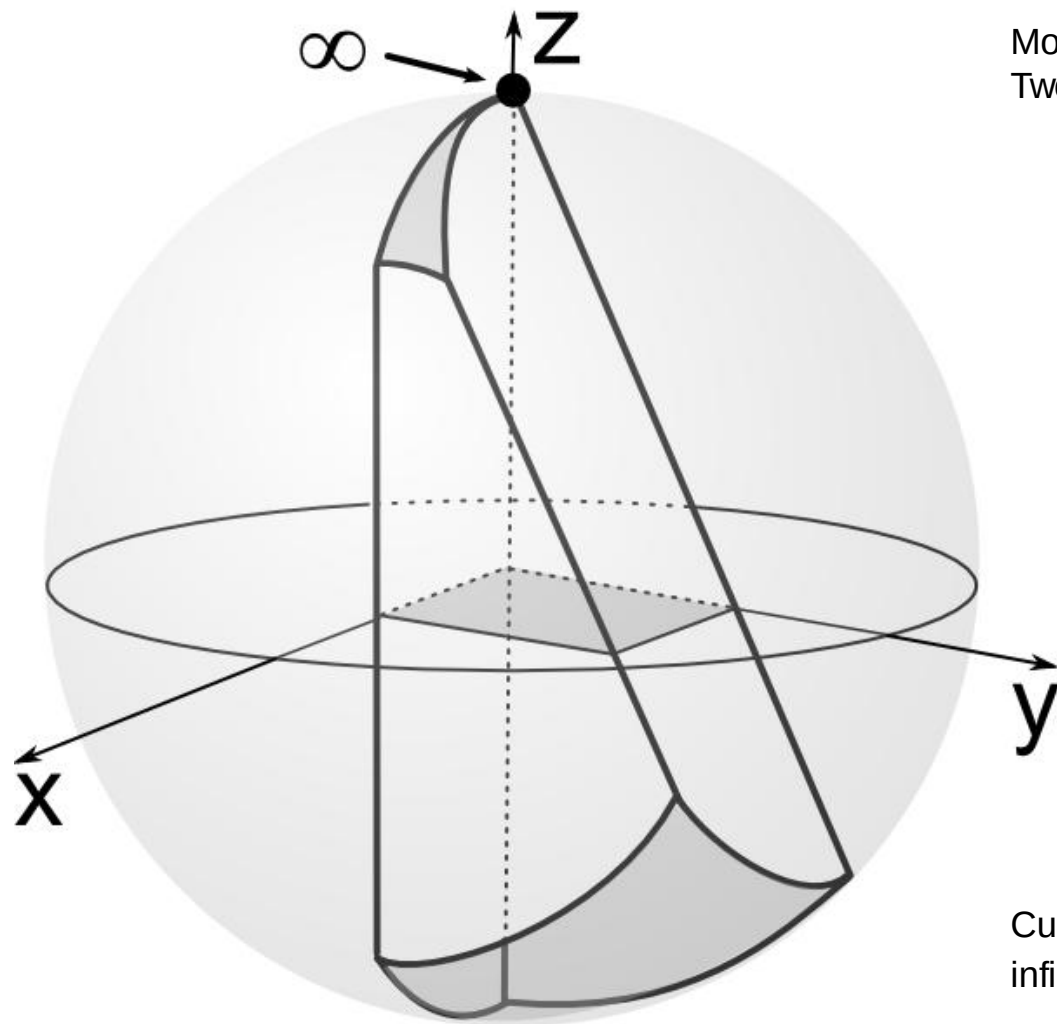
Bending *2223 kaleidoscope



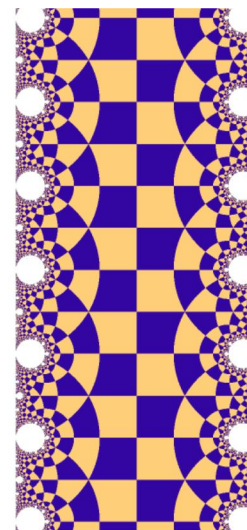
Small tilt.
Disk is transformed into shape with fractal boundary.



Bending *2223 kaleidoscope

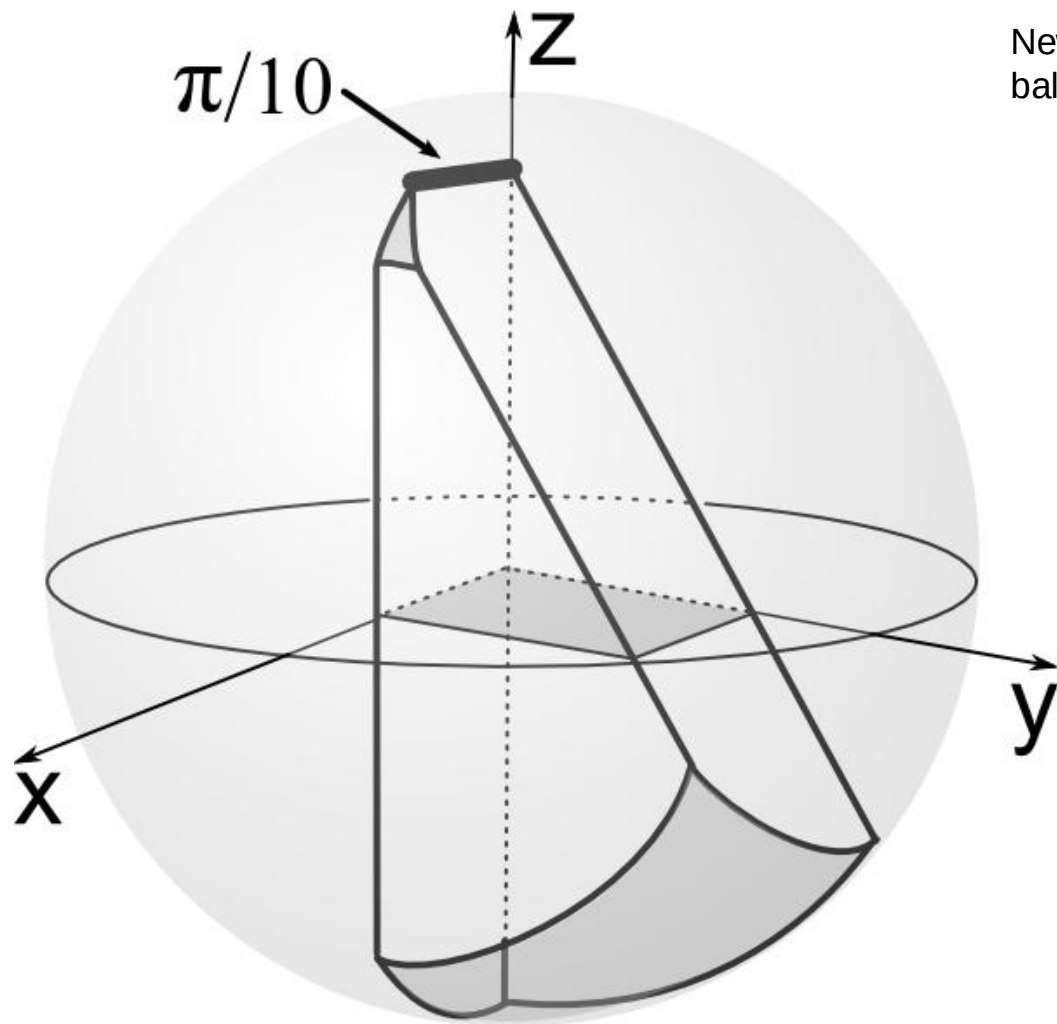


More tilt.
Two sides touch at infinity and make cusp.

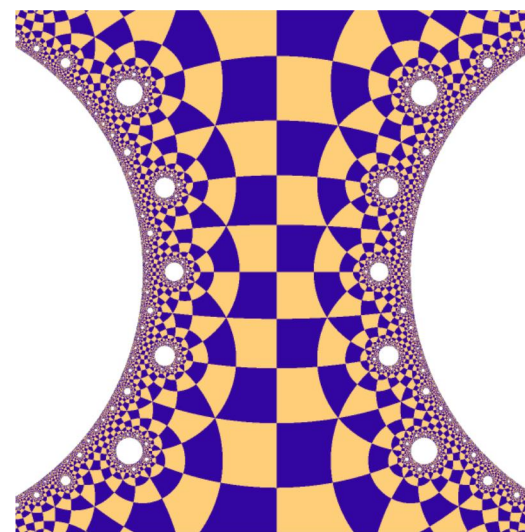


Cusp location at $(0, 0, 1)$ causes tiling to expand to infinity.

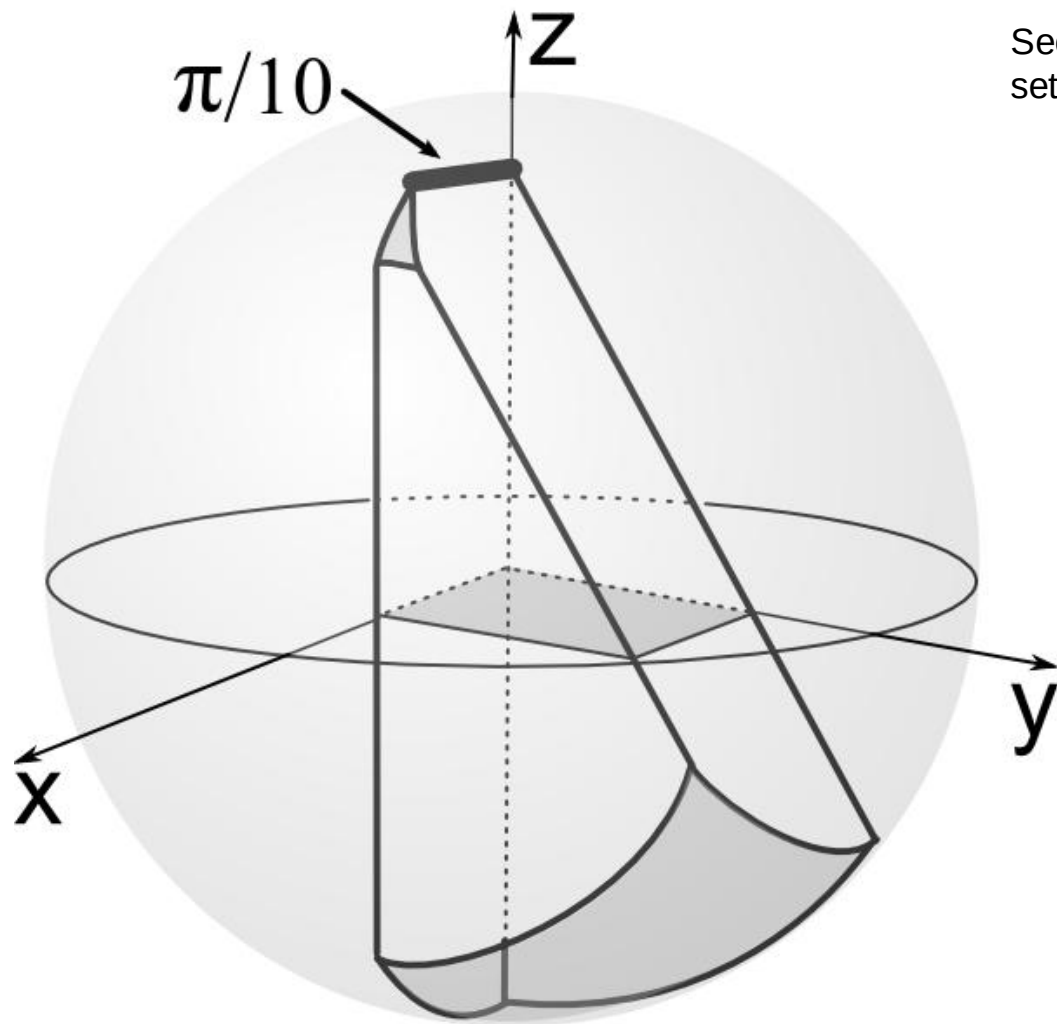
Bending *2223 kaleidoscope



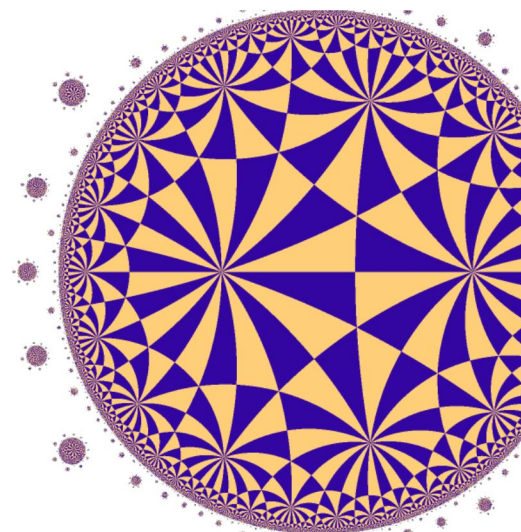
New edge of the "chimney" is partially inside of the unit ball.



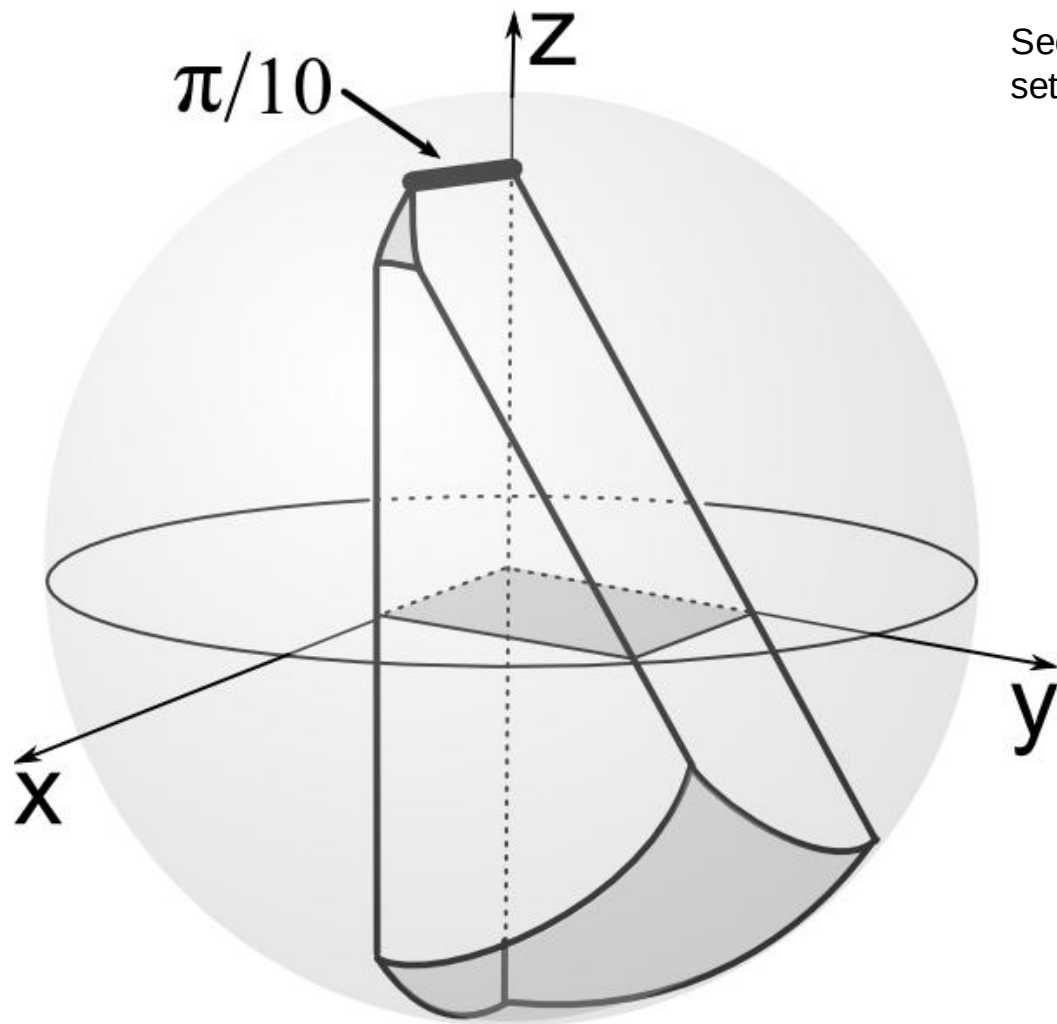
Bending $*2223$ kaleidoscope



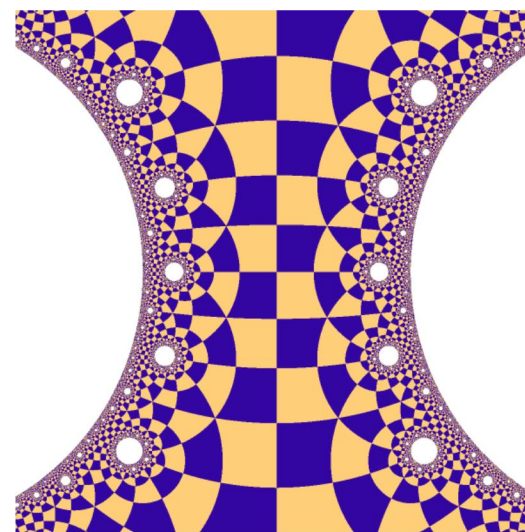
Second component of kaleidoscope bursts into fractal set of $*2,3,10$ triangular kaleidoscopes.



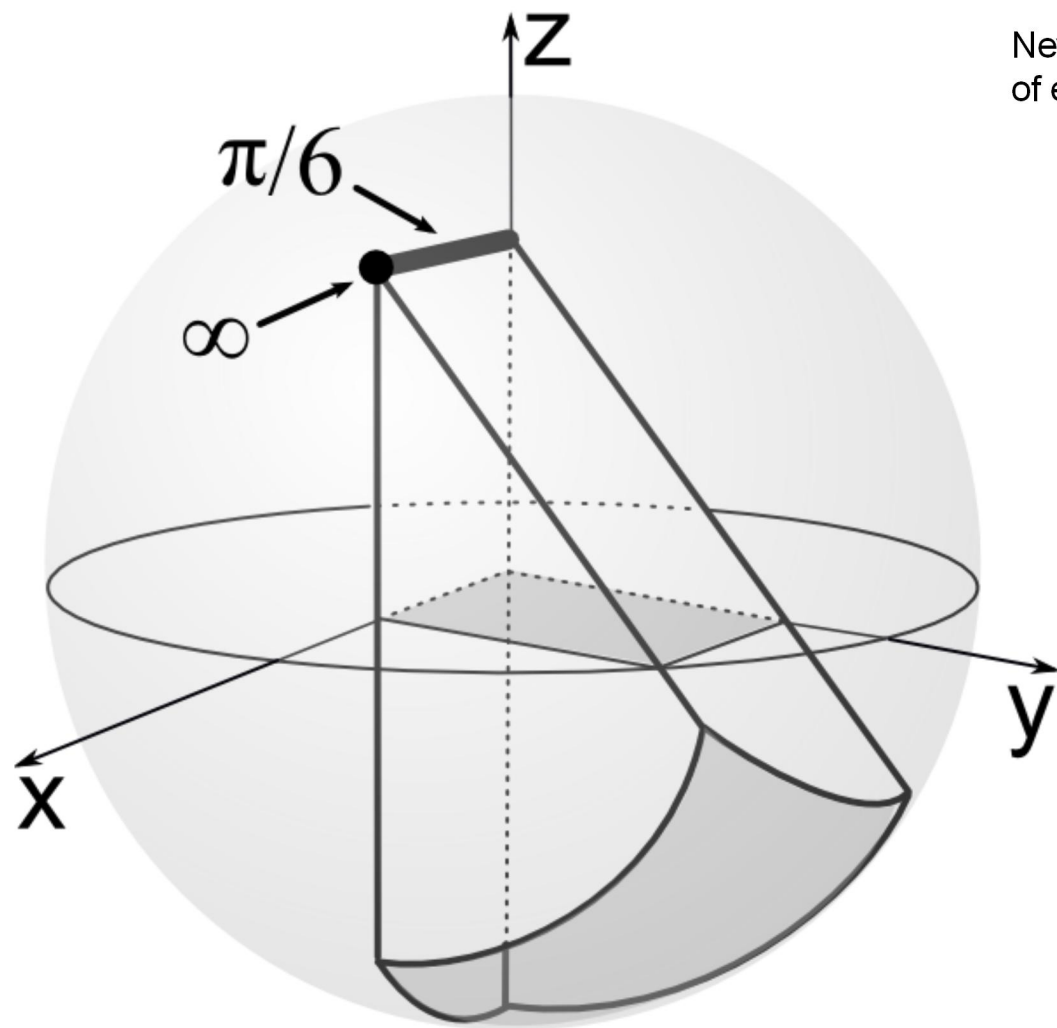
Bending *2223 kaleidoscope



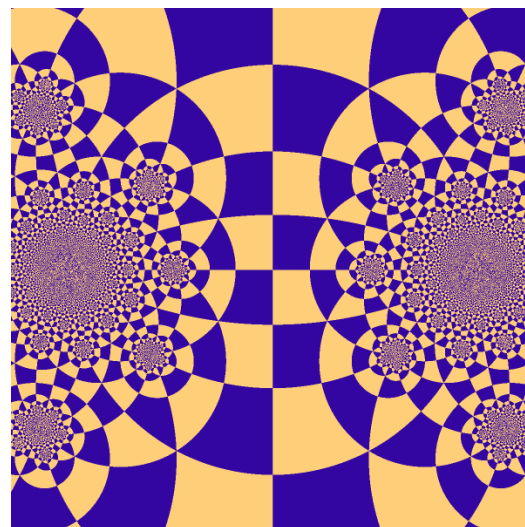
Second component of kaleidoscope bursts into fractal set of *2,3,10 triangular kaleidoscopes.



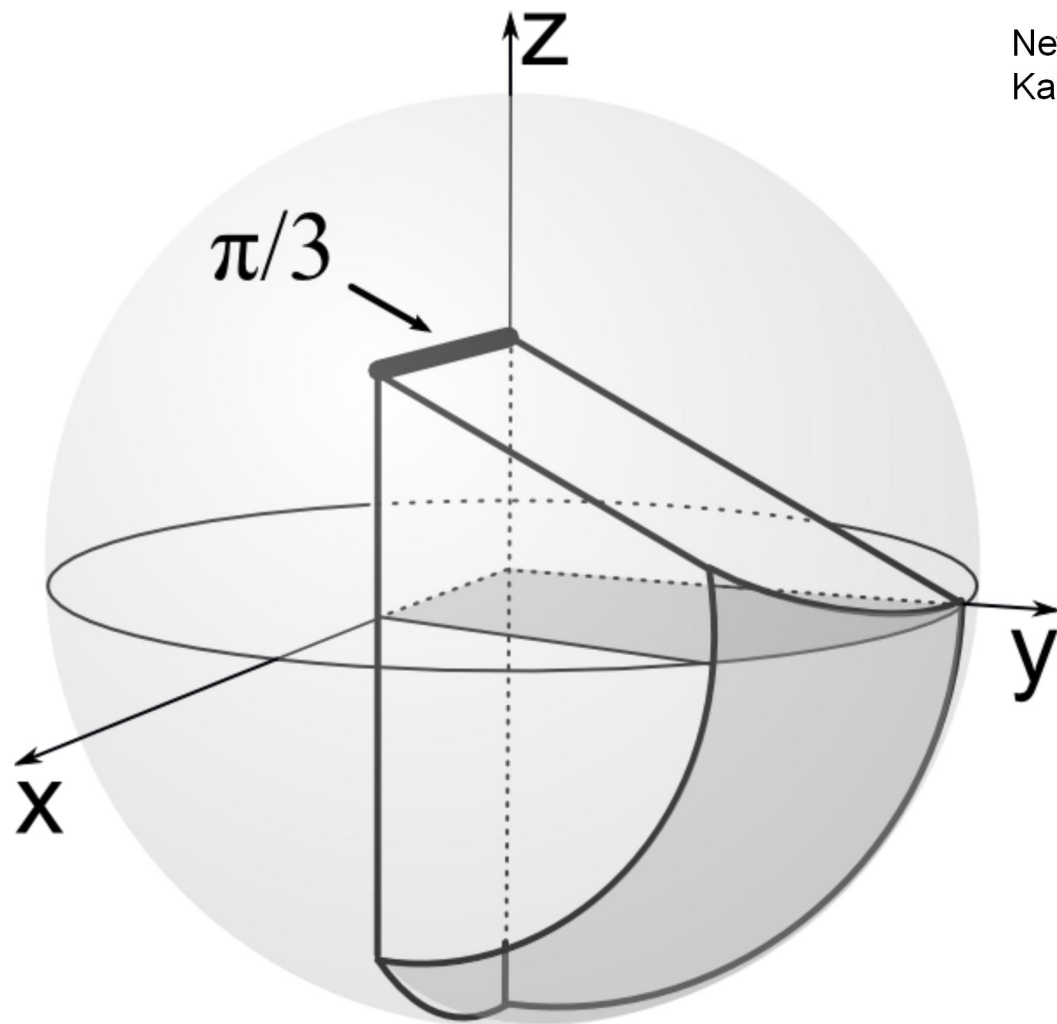
Bending $*2223$ kaleidoscope



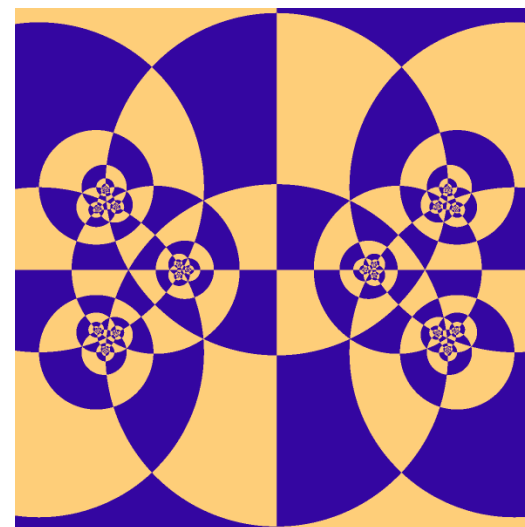
New vertex is formed at infinity. It generates fractal set of euclidean $*236$ kaleidoscopes.



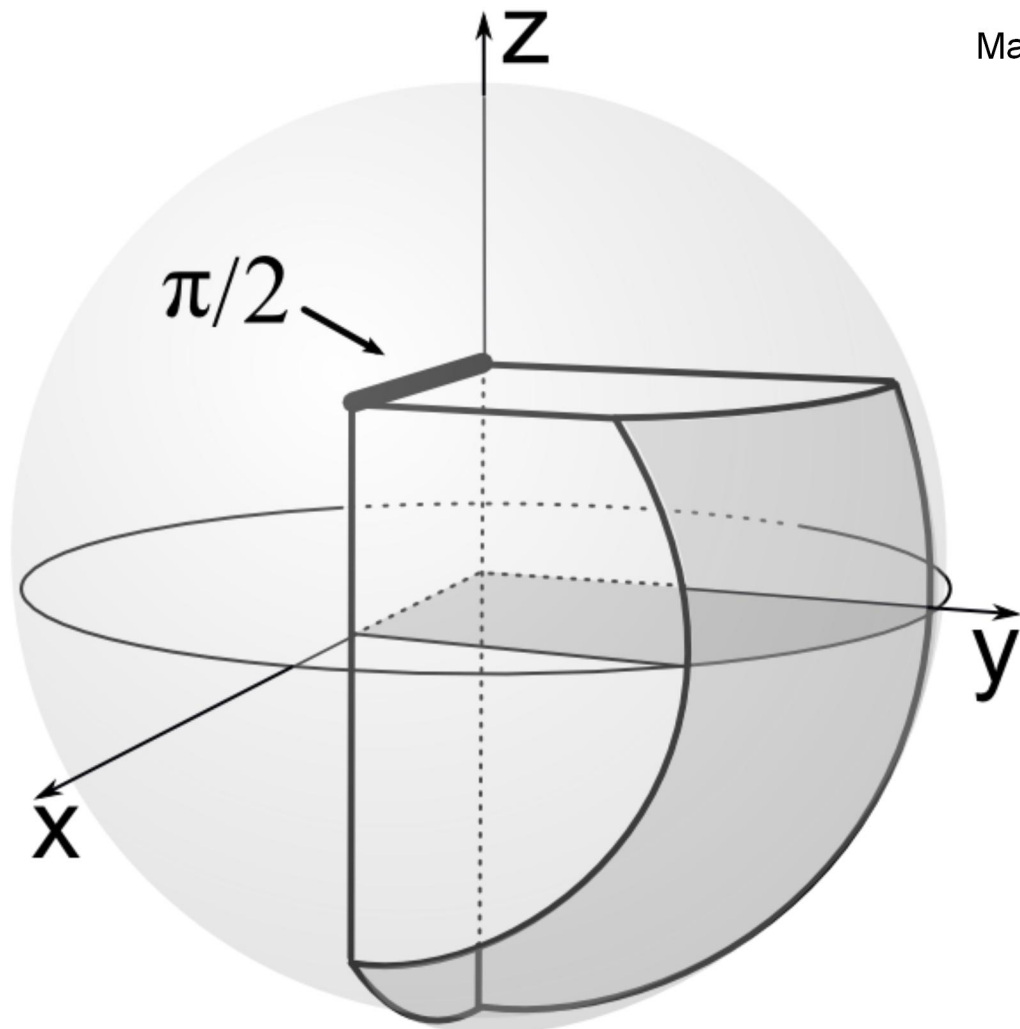
Bending *2223 kaleidoscope



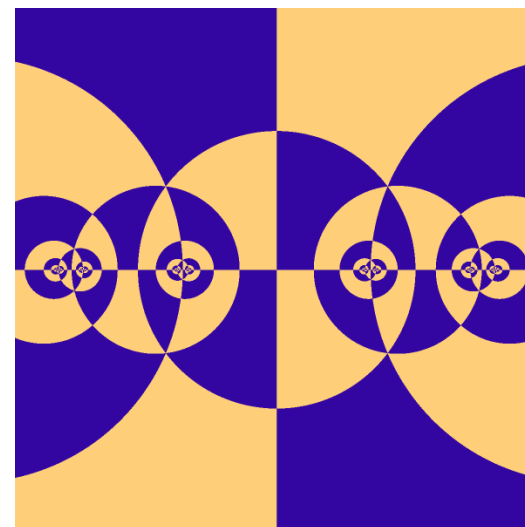
New vertex is moved inside of Poincare ball.
Kaleidoscope has now one component.



Bending *2223 kaleidoscope



Maximal bend of *2223 kaleidoscope.



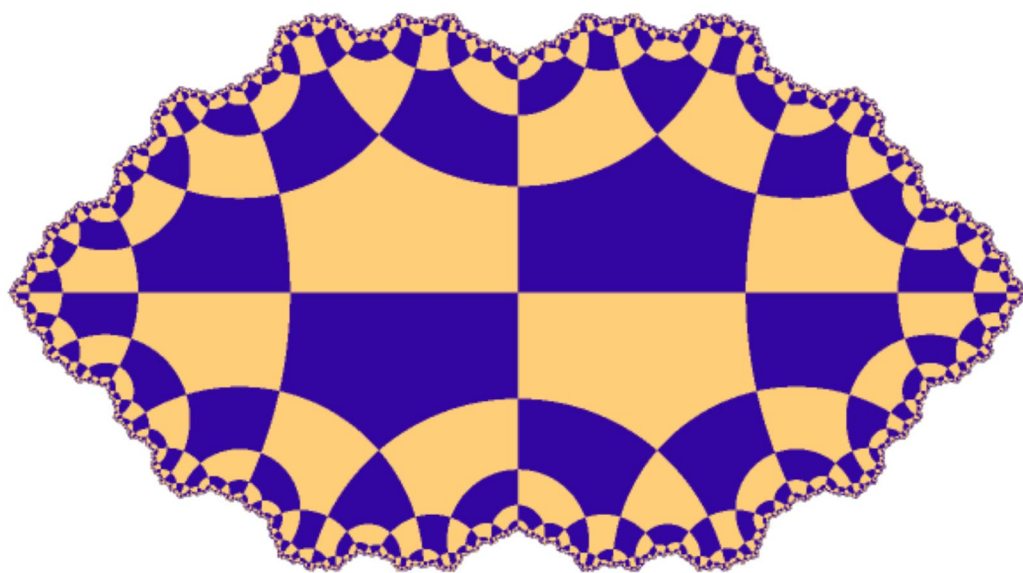
Bending n-gonal kaleidoscopes

Any n-gonal kaleidoscope can be bend

n-gonal kaleidoscope has $2(n-3)$
independent continuous parameters

$(n-3)$ stretching parameters

$(n-3)$ bending parameters



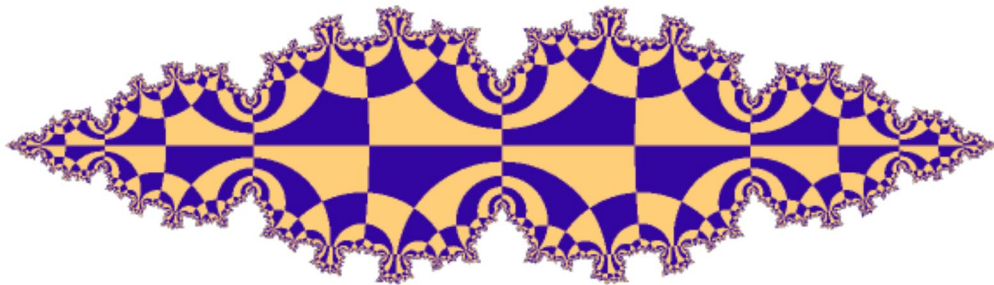
Bending n-gonal kaleidoscopes

Any n-gonal kaleidoscope can be bend

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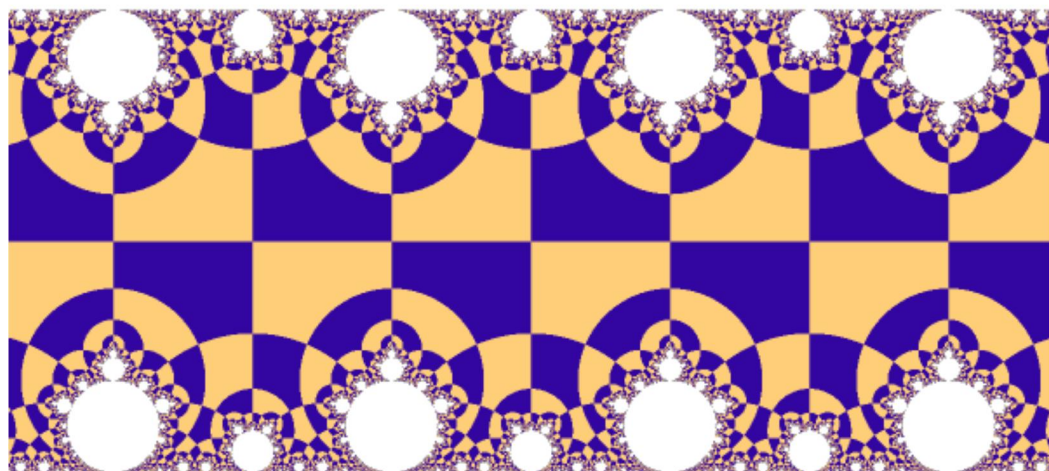
Bending n-gonal kaleidoscopes

Any n-gonal kaleidoscope can be bend

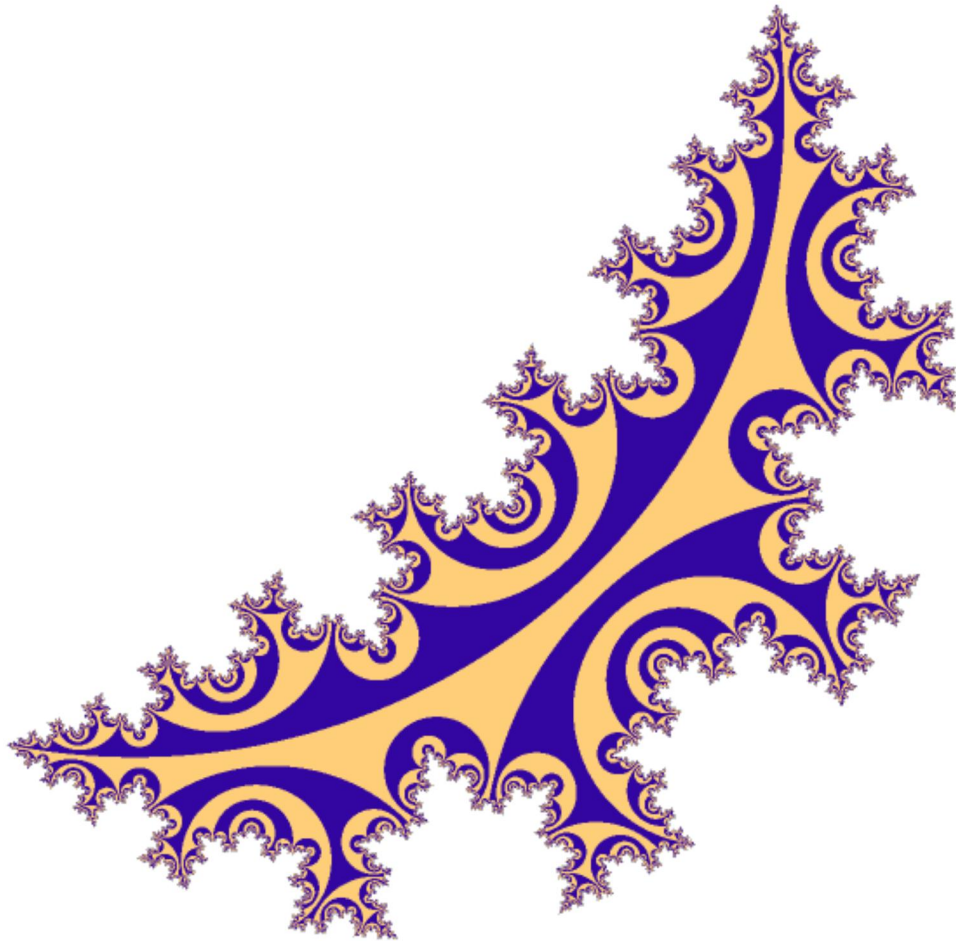
n-gonal kaleidoscope has $2(n-3)$
independent continuous parameters

$(n-3)$ stretching parameters

$(n-3)$ bending parameters



Bending n-gonal kaleidoscopes



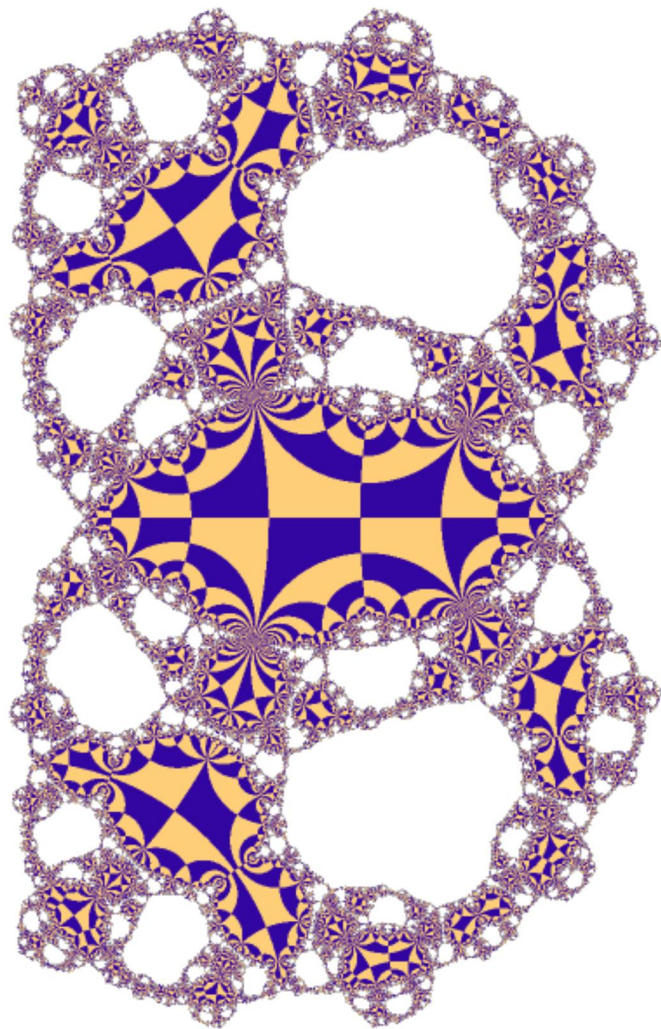
Any n -gonal kaleidoscope can be bend

n -gonal kaleidoscope has $2(n-3)$
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$(n-3)$ stretching parameters

$(n-3)$ bending parameters

Bending n-gonal kaleidoscopes



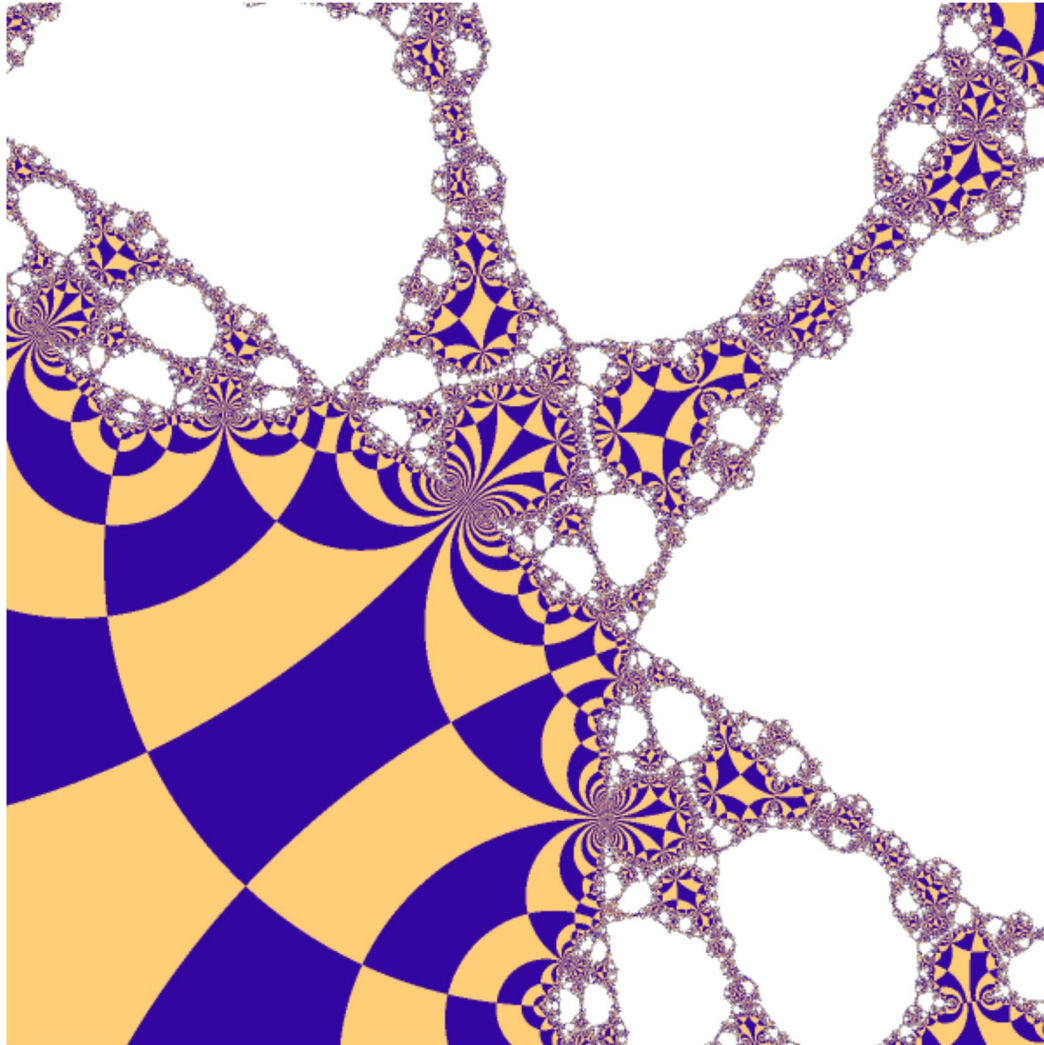
Any n -gonal kaleidoscope can be bend

n -gonal kaleidoscope has $2(n-3)$
independent continuous parameters

$(n-3)$ stretching parameters

$(n-3)$ bending parameters

Bending n-gonal kaleidoscopes



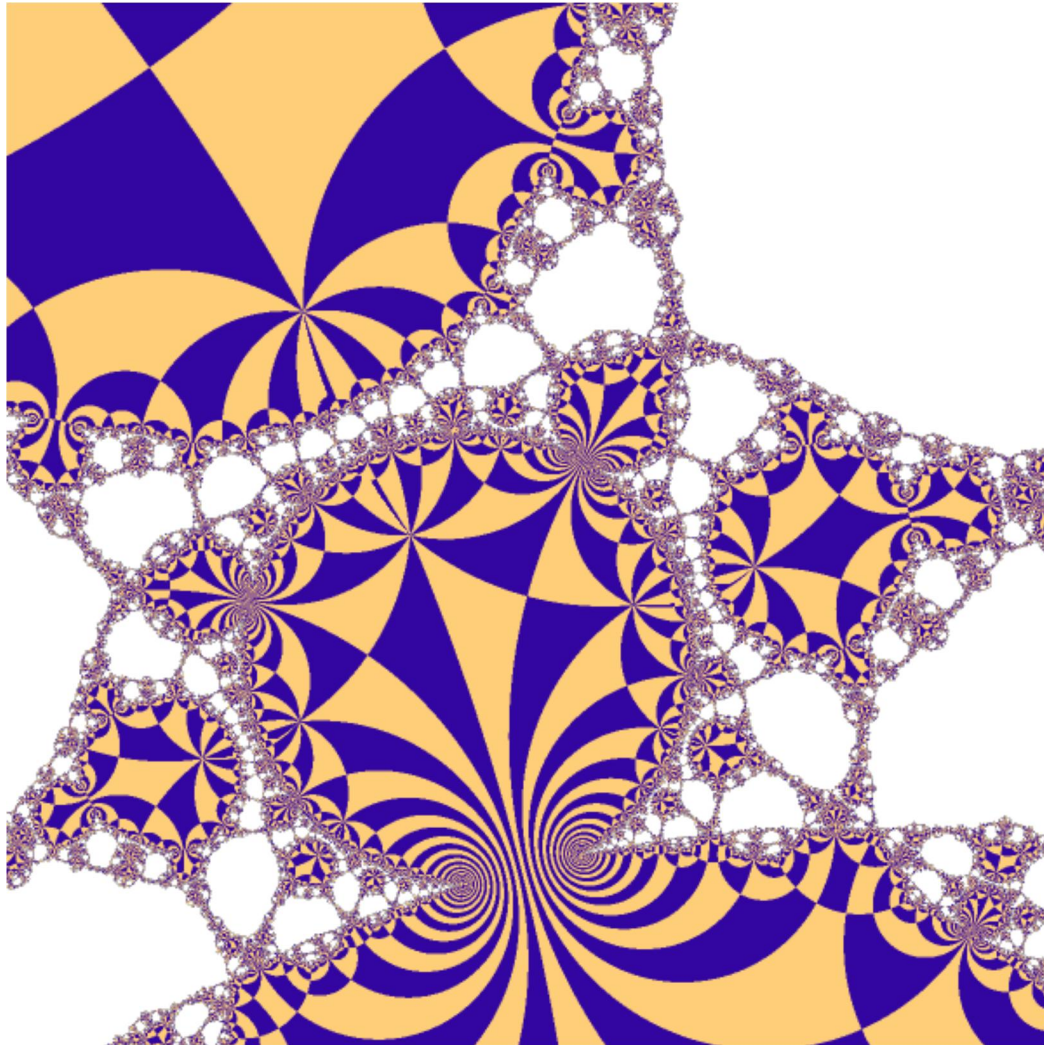
Any n -gonal kaleidoscope can be bent

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independent continuous parameters

$(n-3)$ stretching parameters

$(n-3)$ bending parameters

Bending n-gonal kaleidoscopes



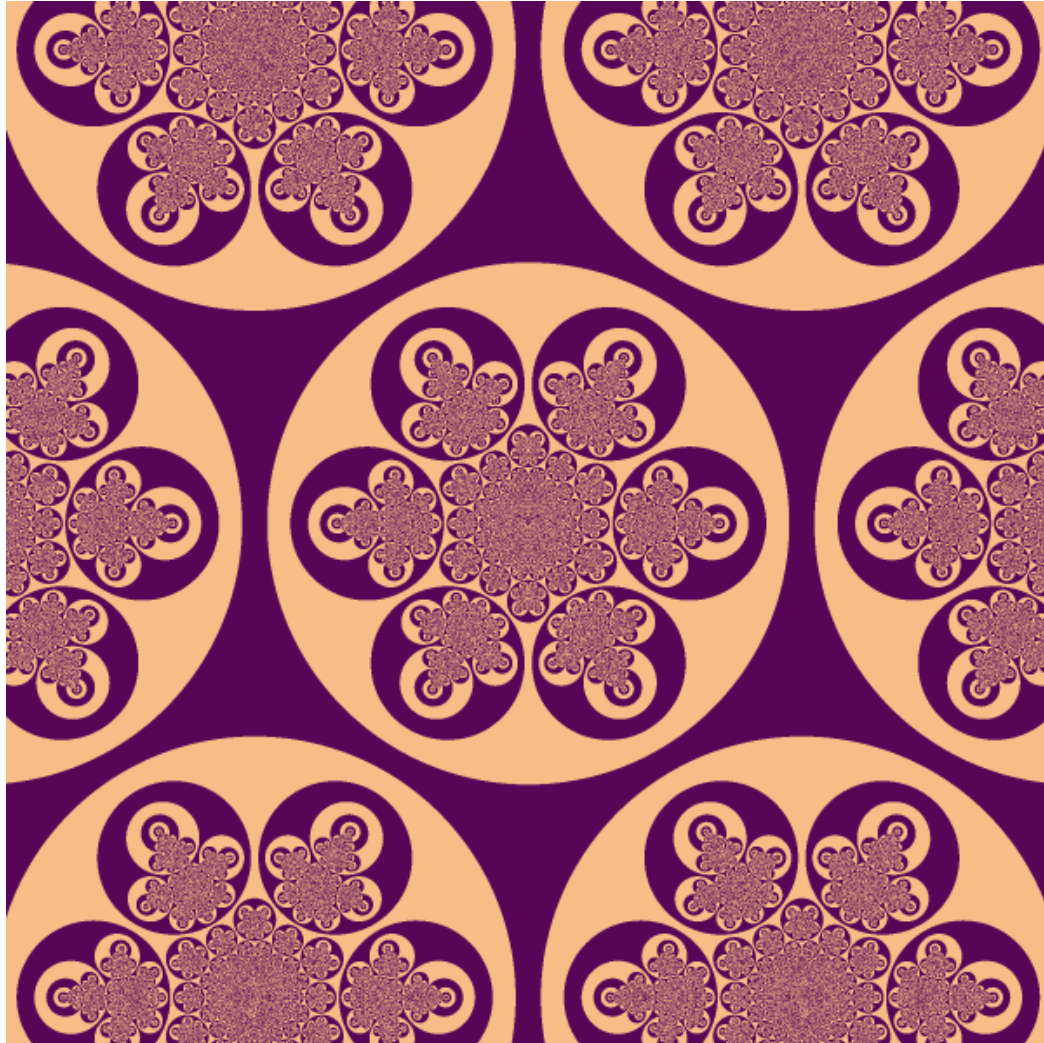
Any n -gonal kaleidoscope can be bent

n -gonal kaleidoscope has $2(n-3)$ independent continuous parameters

$(n-3)$ stretching parameters

$(n-3)$ bending parameters

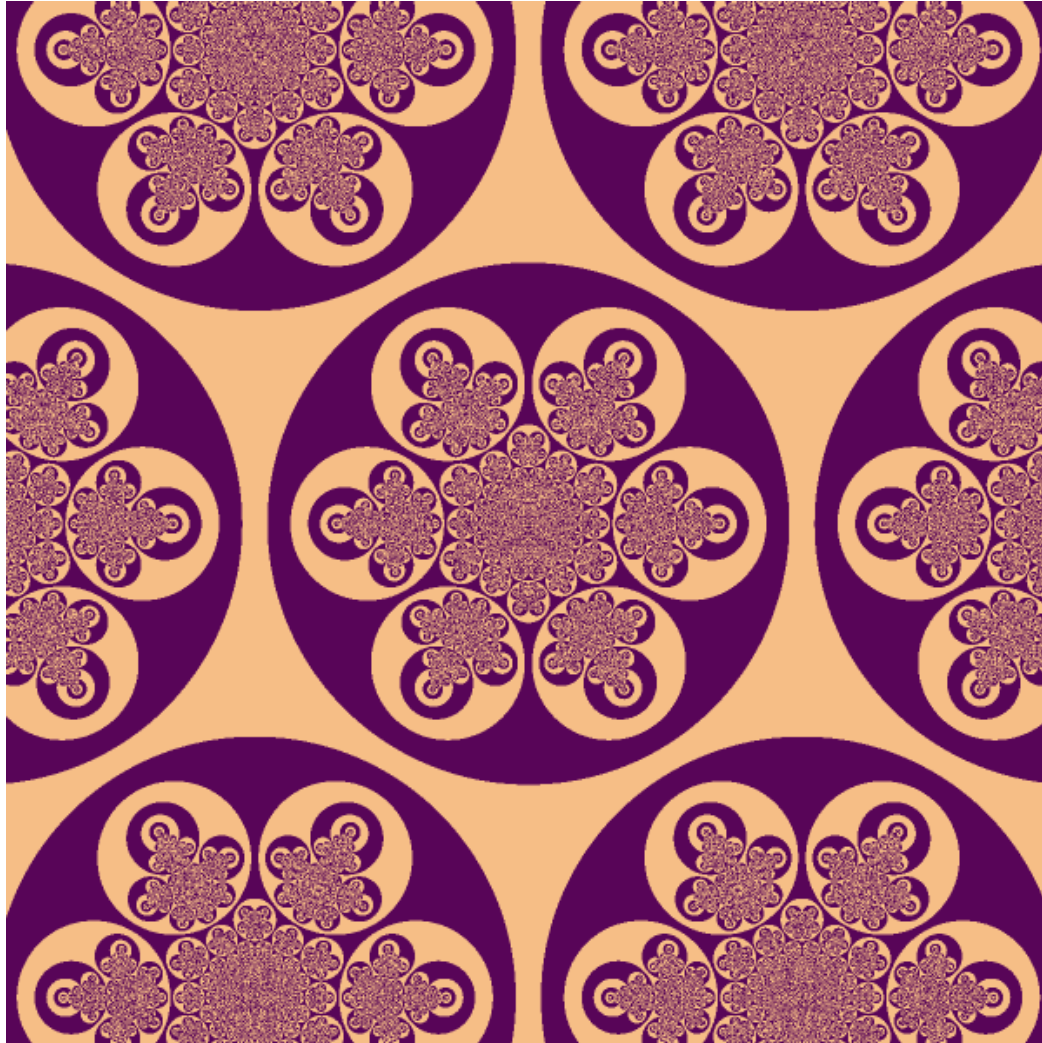
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

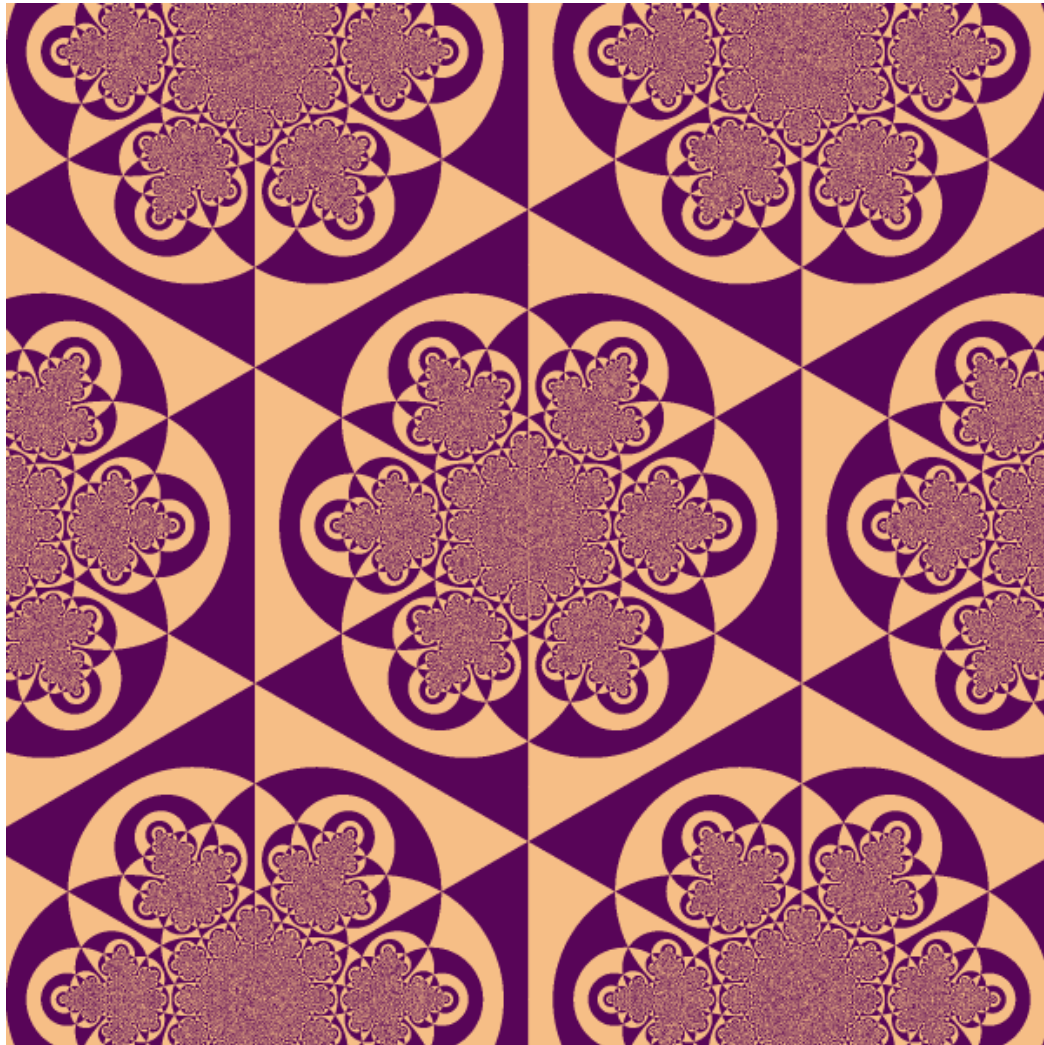
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

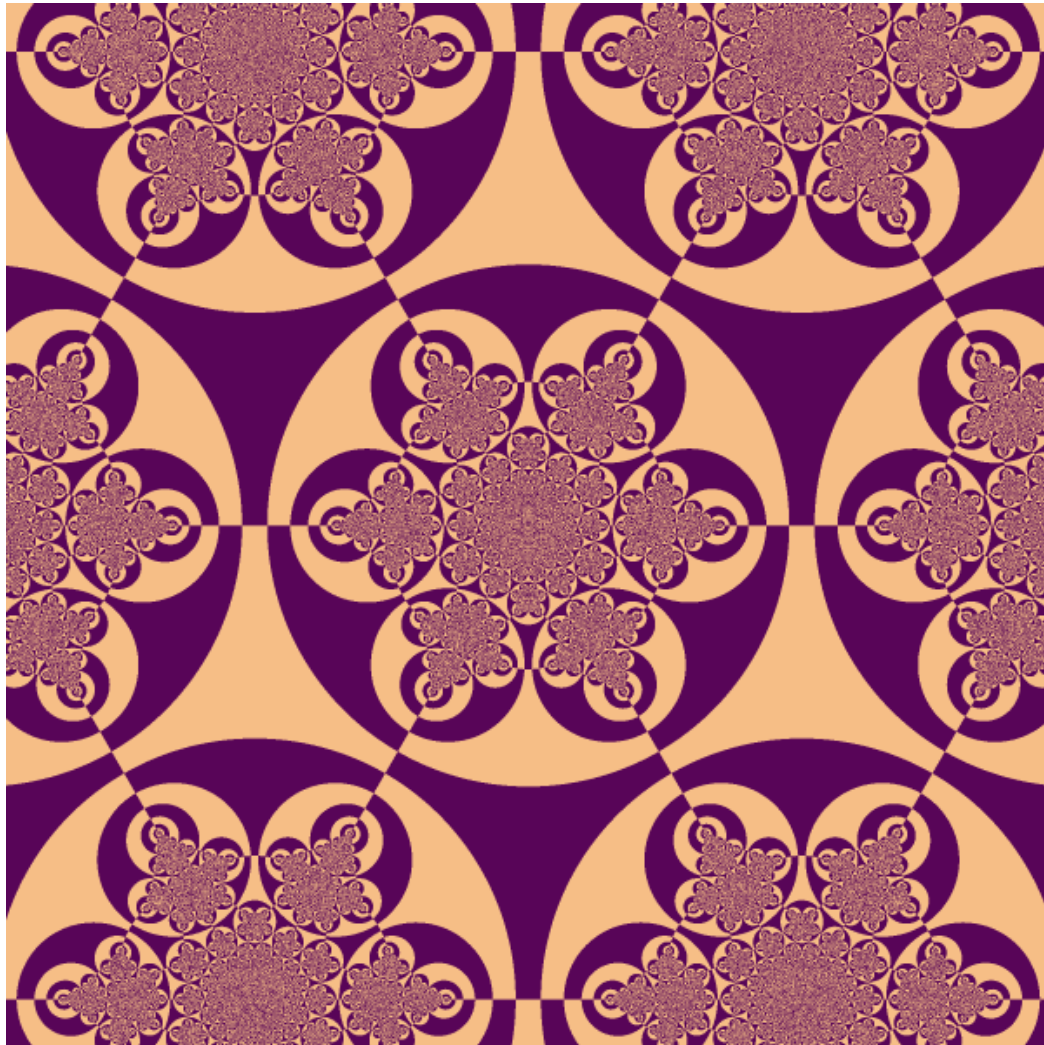
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

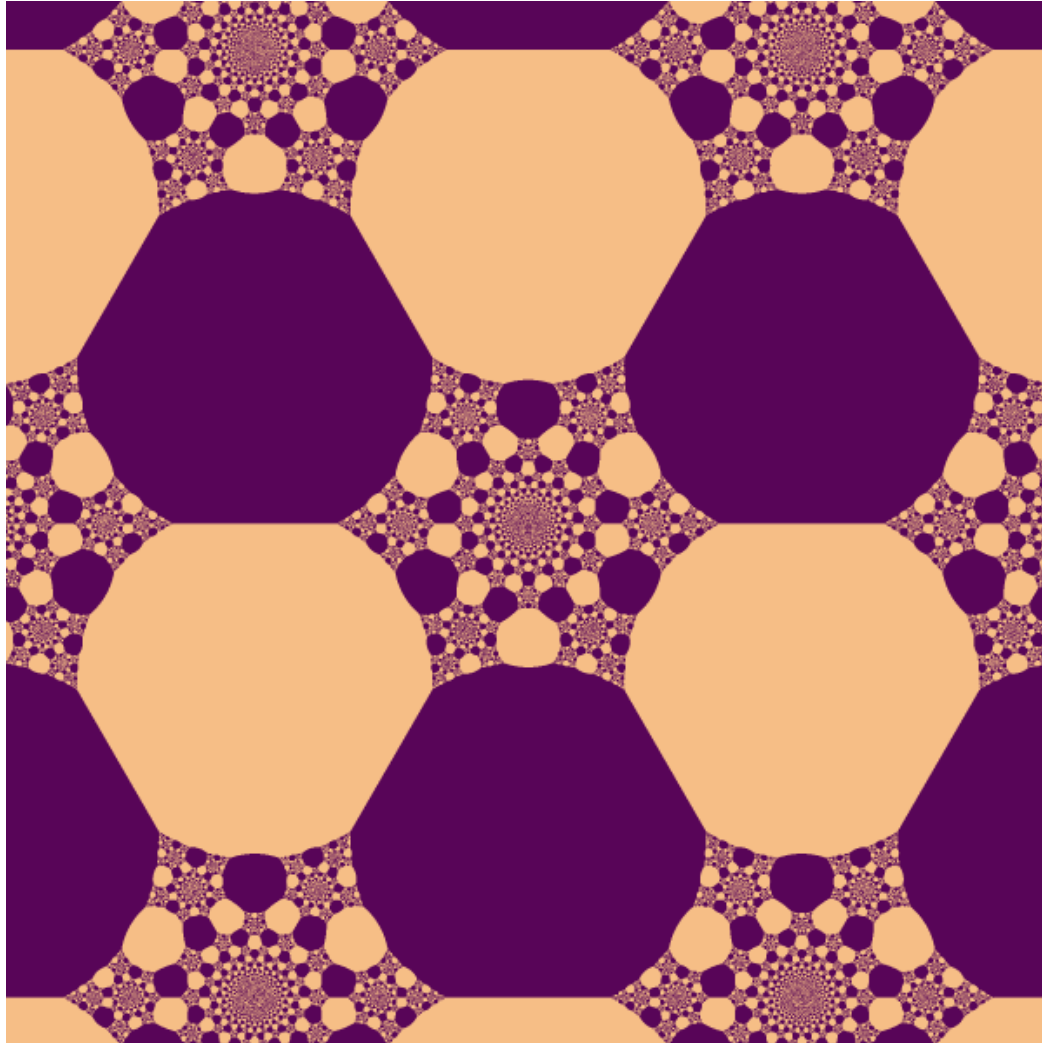
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

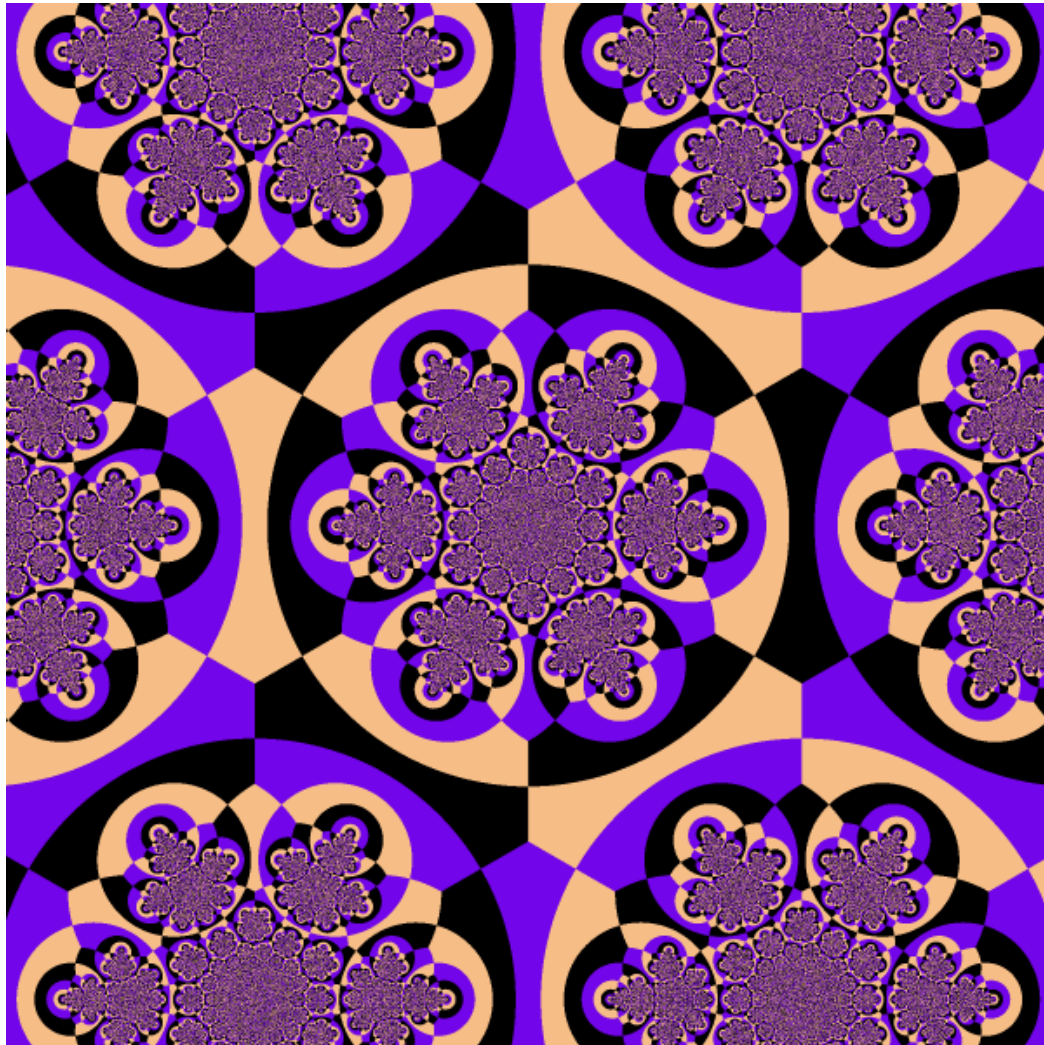
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

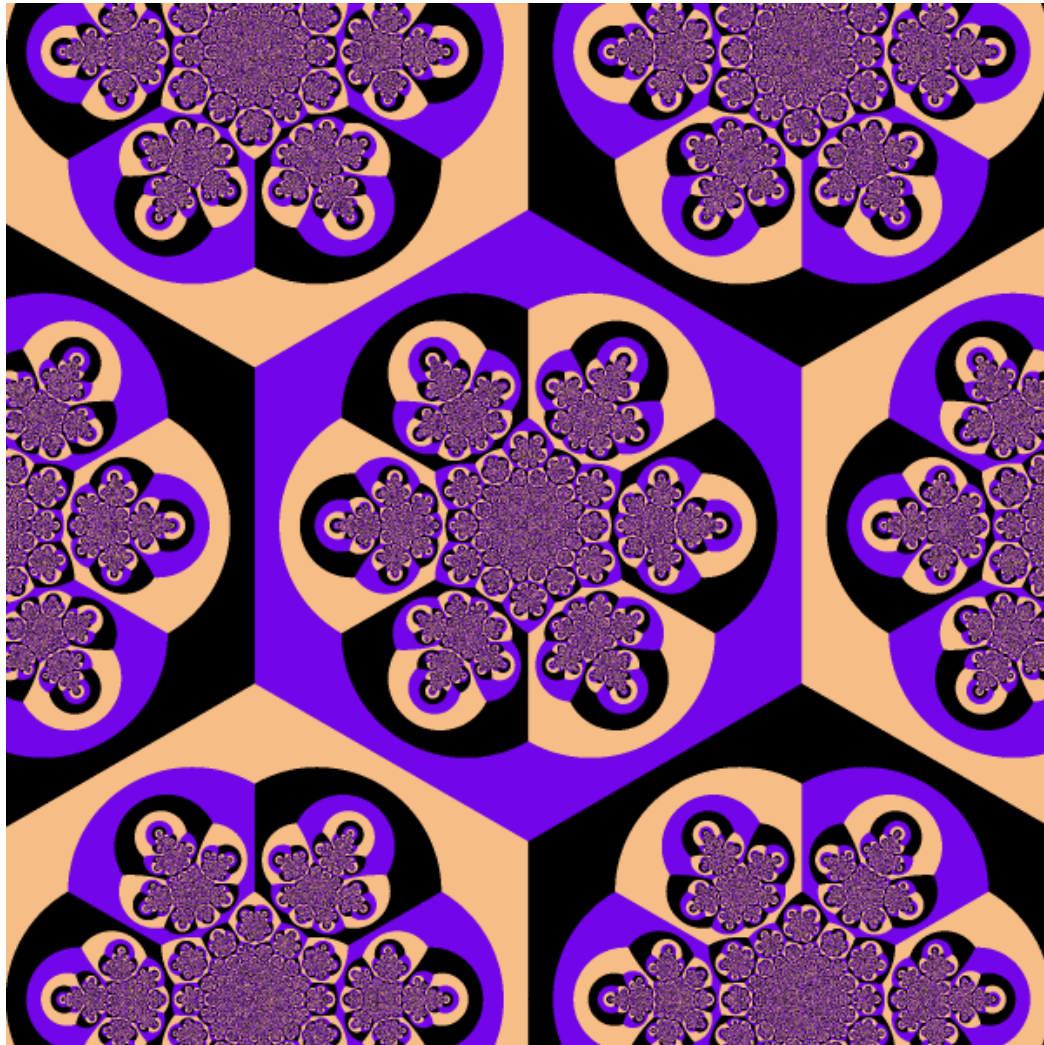
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

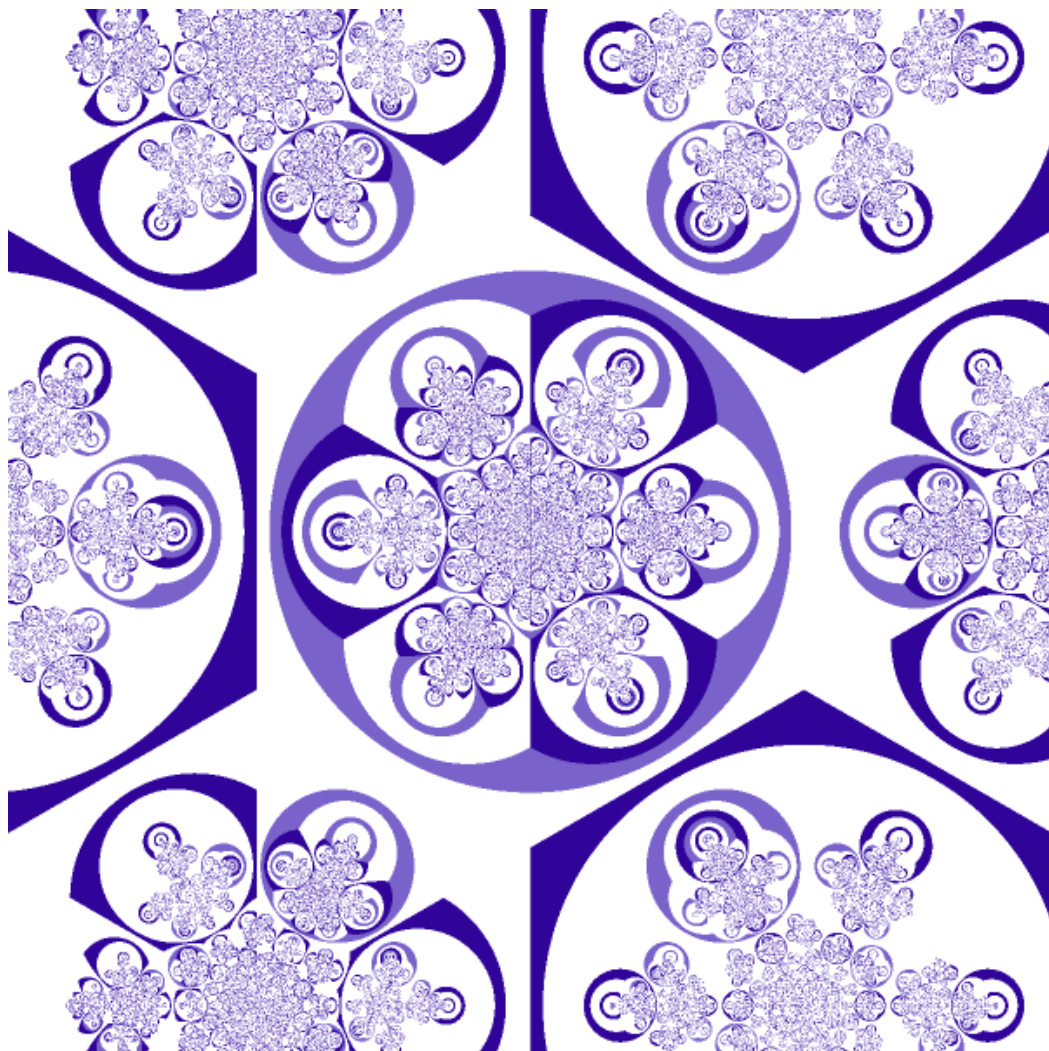
Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup are colored in different colors.

Coloring using cosets



Let's play with color.

Tiles from different cosets of a subgroup group are colored in different colors.

Klein-Fricke "Composition Group"

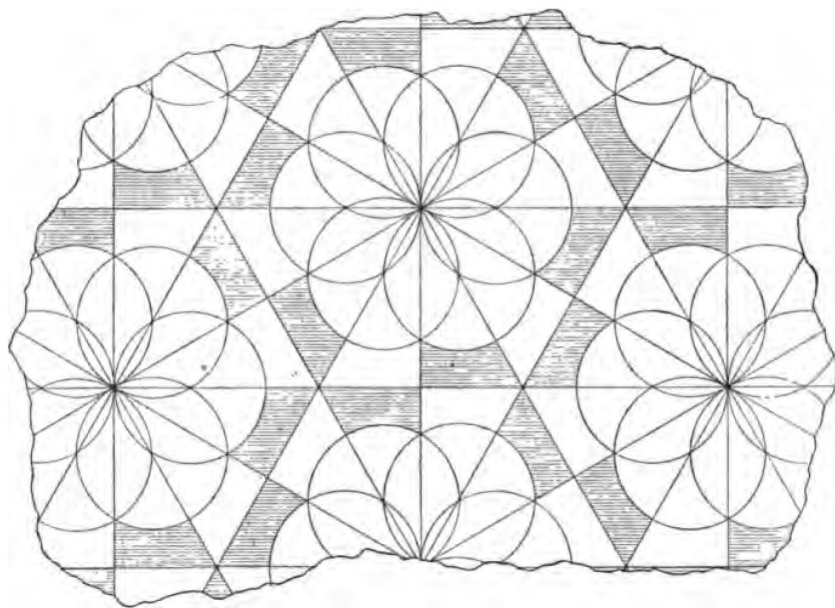
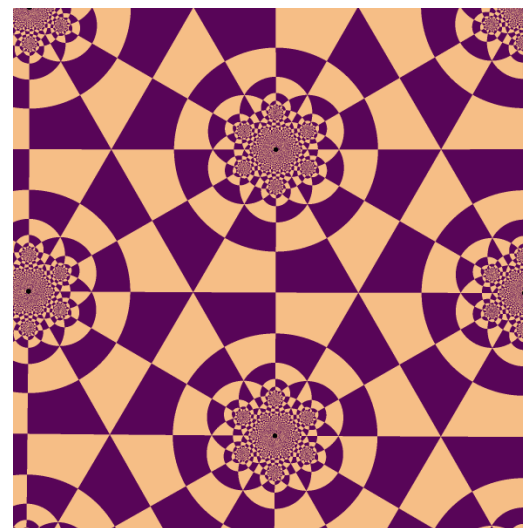


Fig. 152.

Similar $*2323$ kaleidoscope was illustrated before in Klein F., Fricke R. (1897) *Vorlesungen Über die Theorie der Automorphen Functionen*

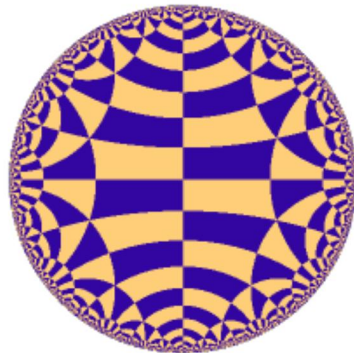
The last series of images was for bended $*2223$ kaleidoscope with $*236$ parabolic subgroup (cusp at infinity).



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

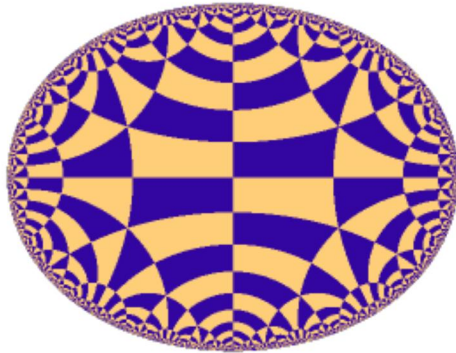
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

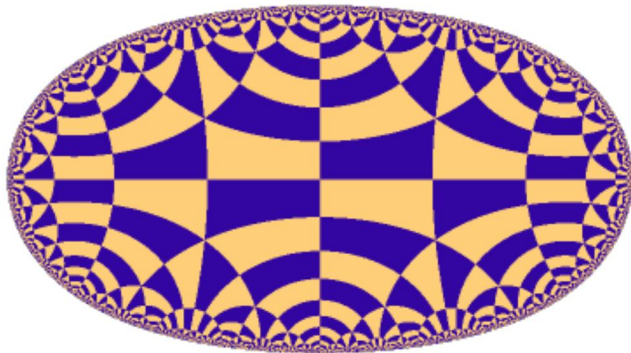
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

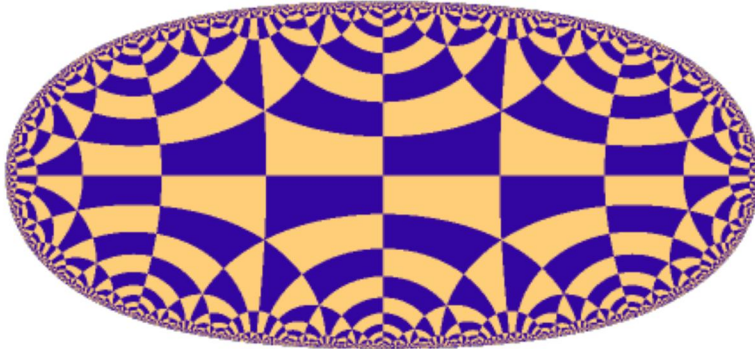
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

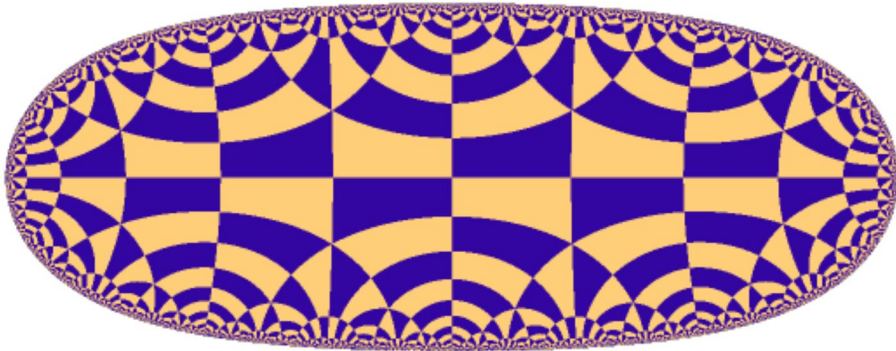
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

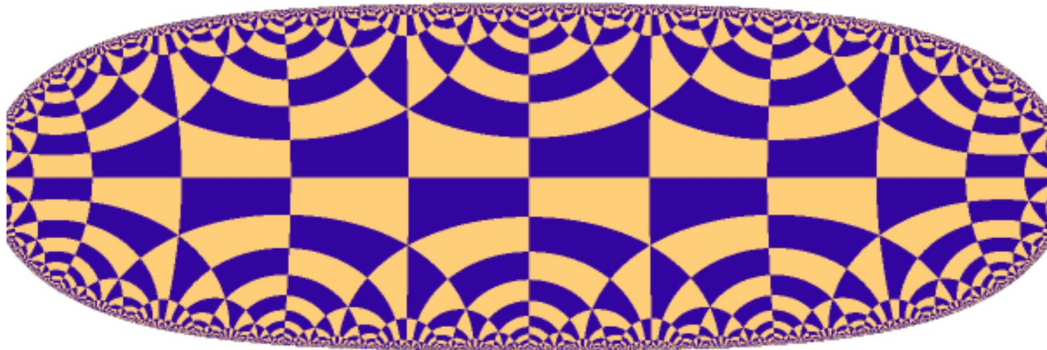
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

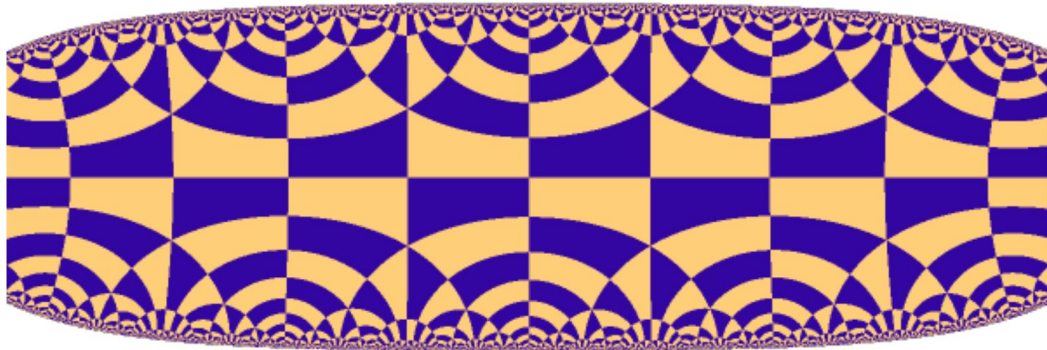
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



Conformal mapping

This conformal mapping transforms unit disk to horizontal band

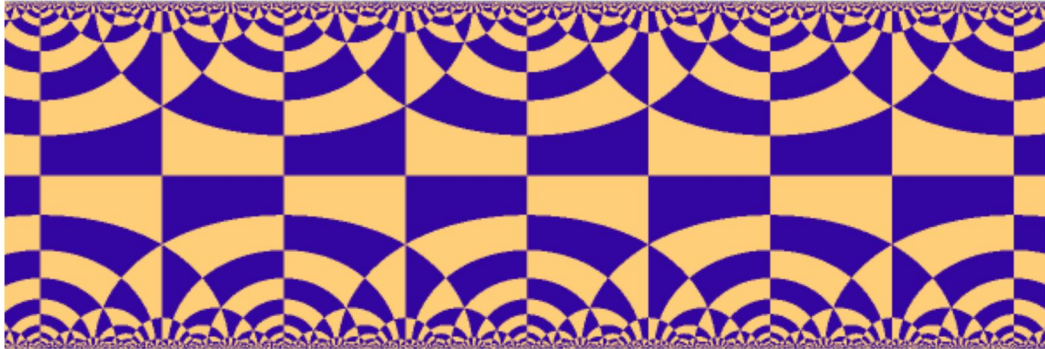
$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$



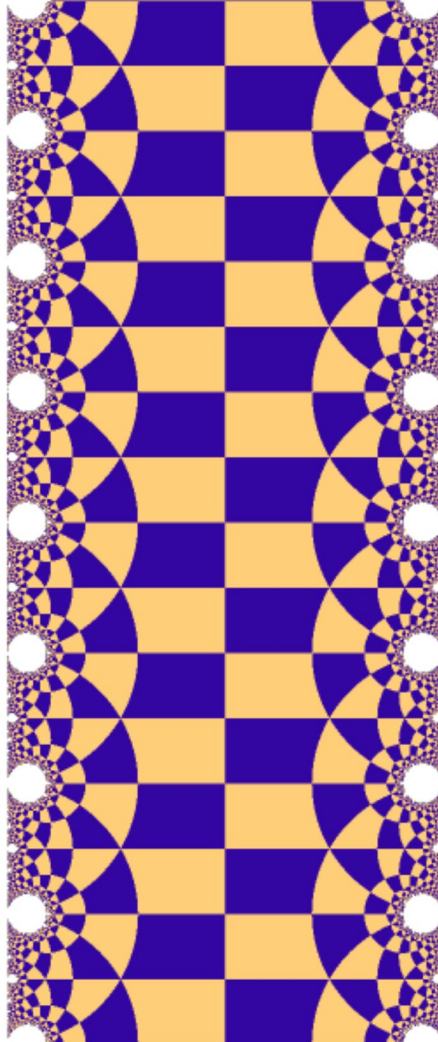
Conformal mapping

This conformal mapping transforms unit disk to horizontal band

$$w = \left(\frac{4}{\pi}\right) \tanh^{-1}(z); z, w \in \mathbb{C}$$

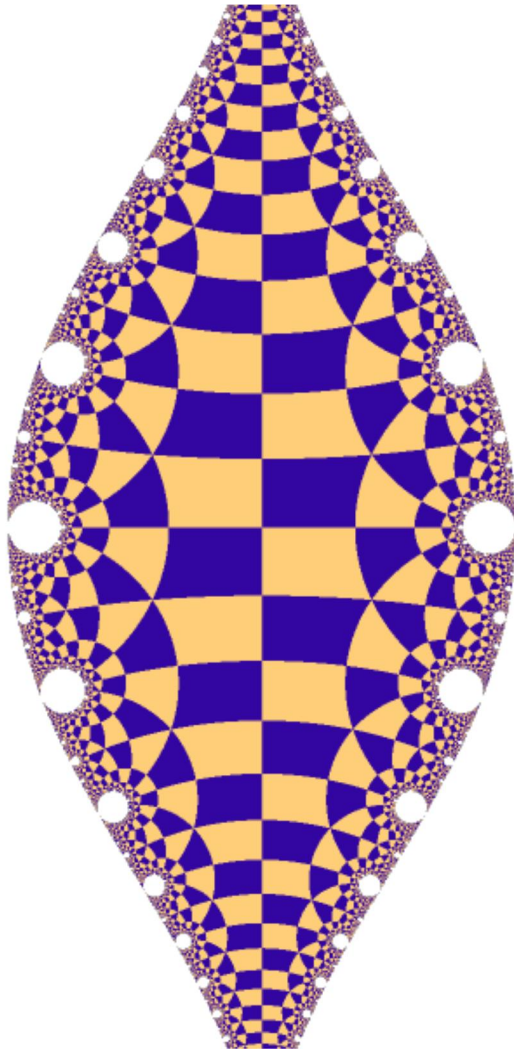


Conformal mapping



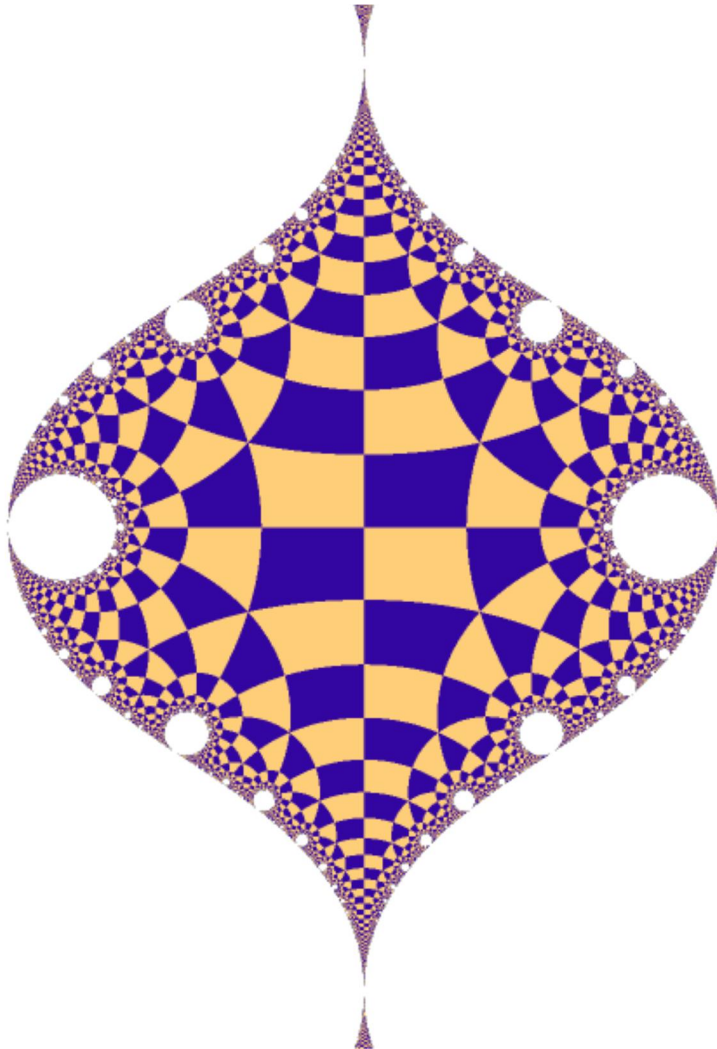
The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping



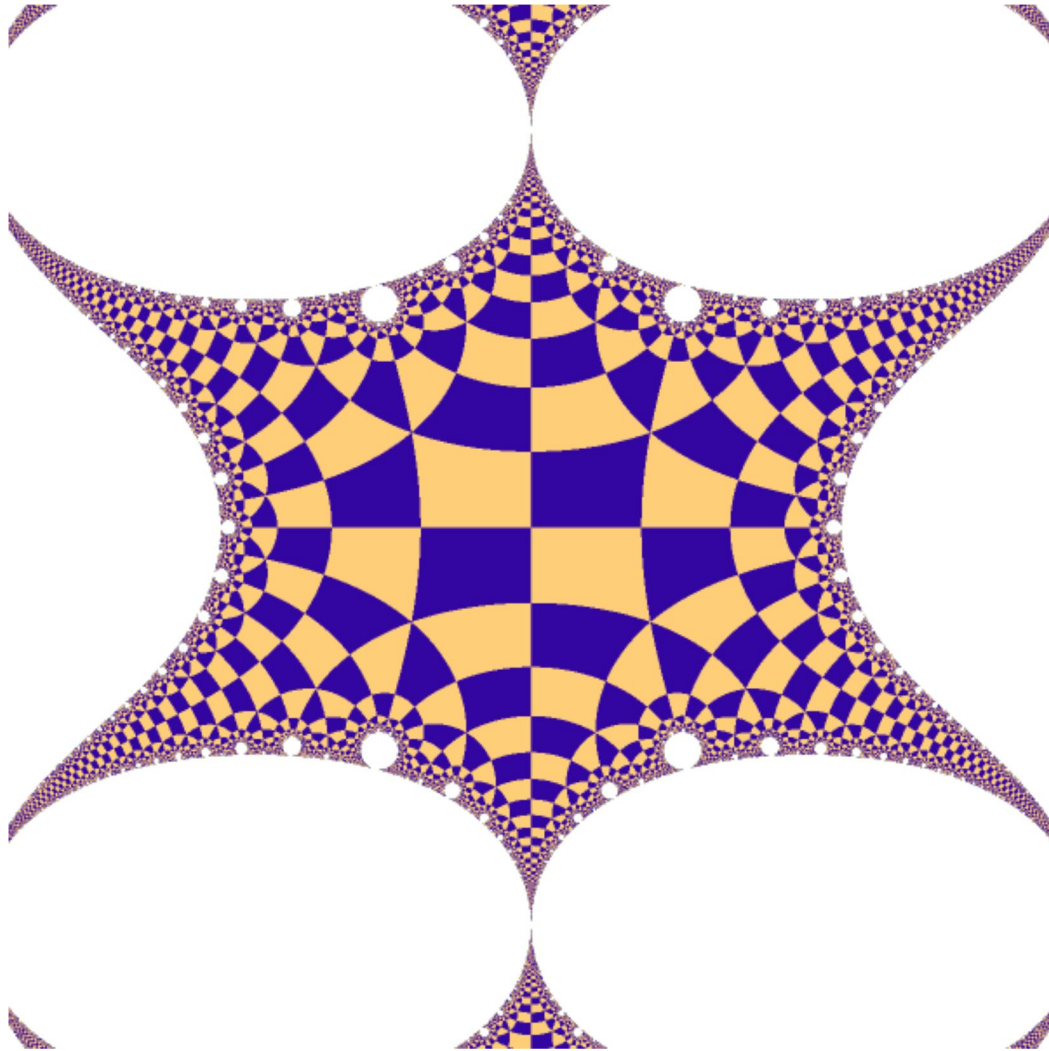
The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping



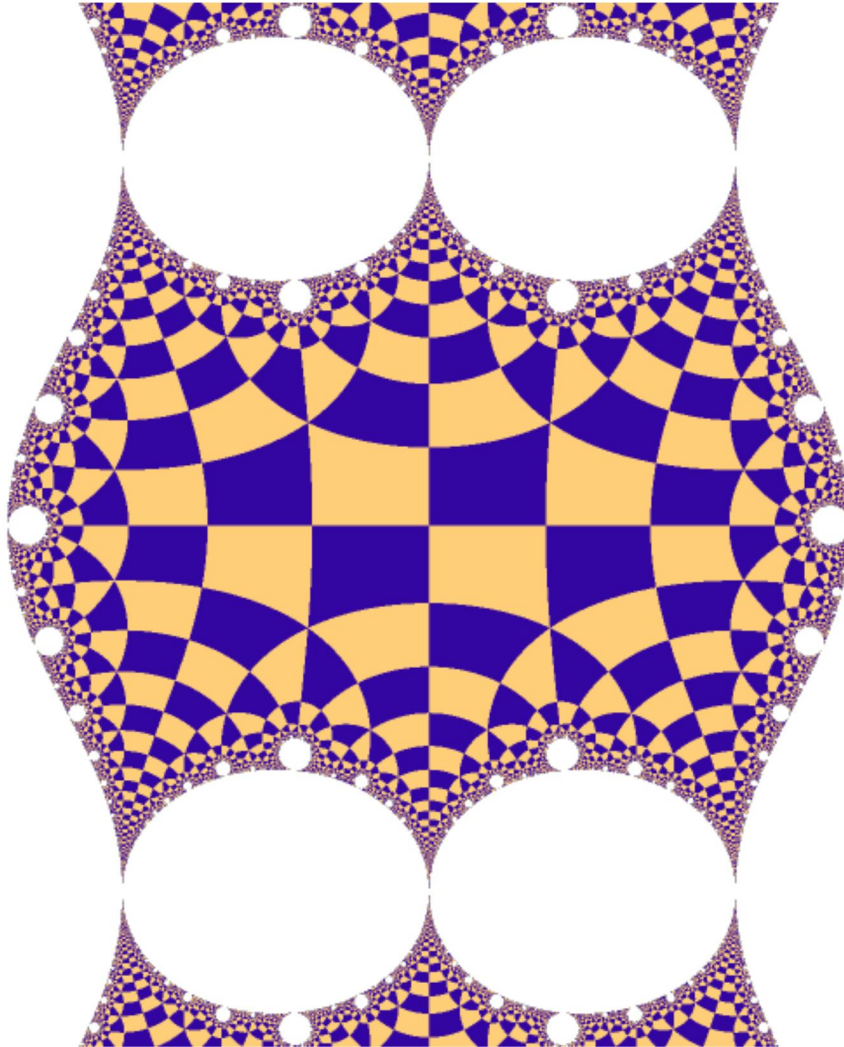
The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping



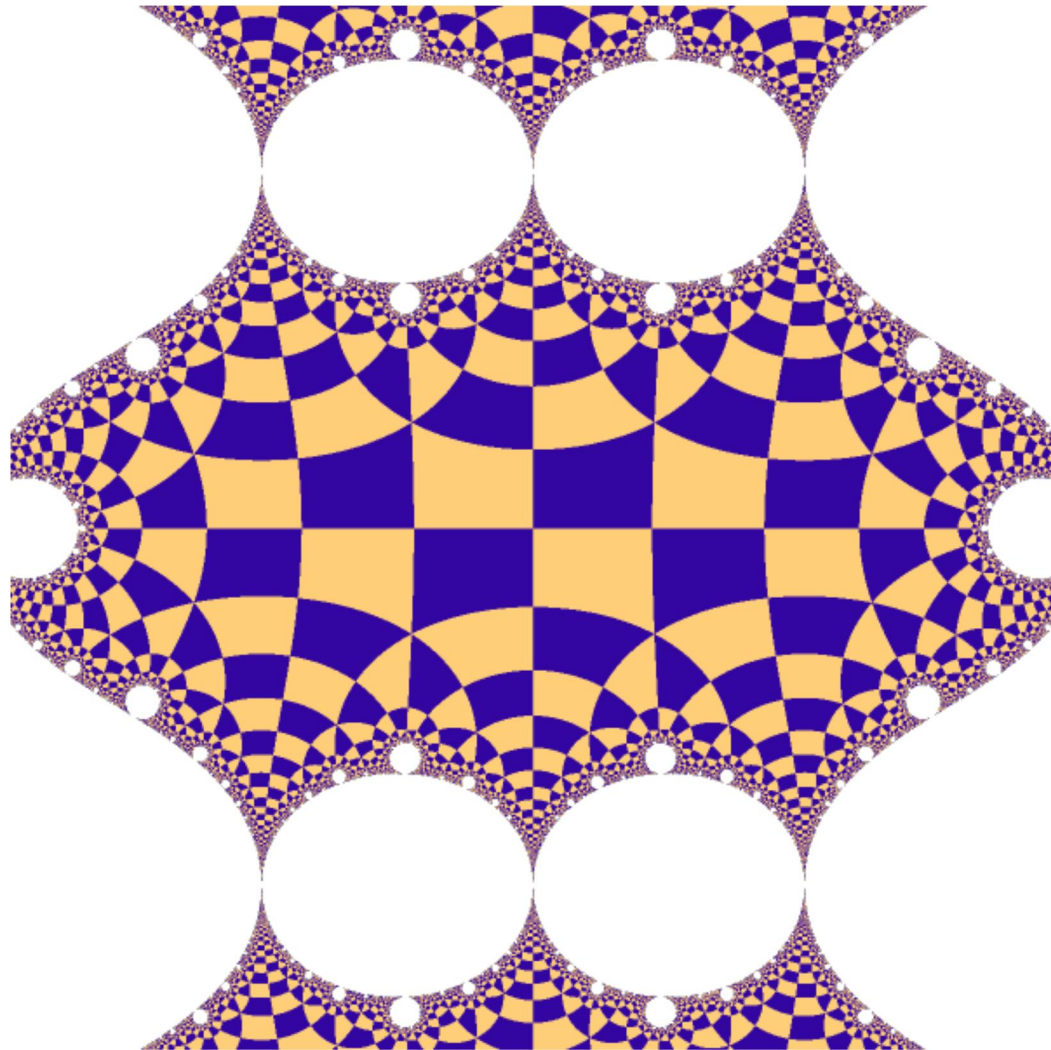
The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping



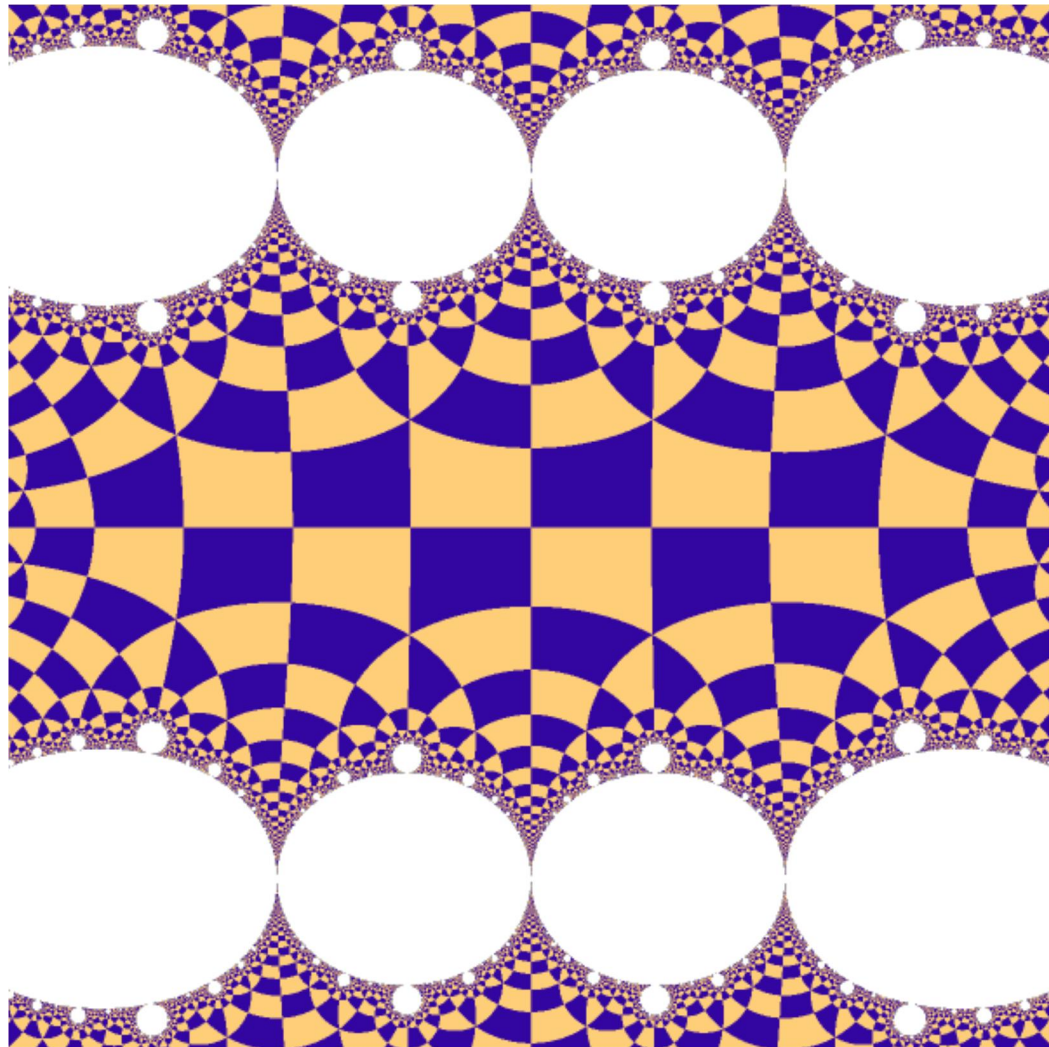
The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping



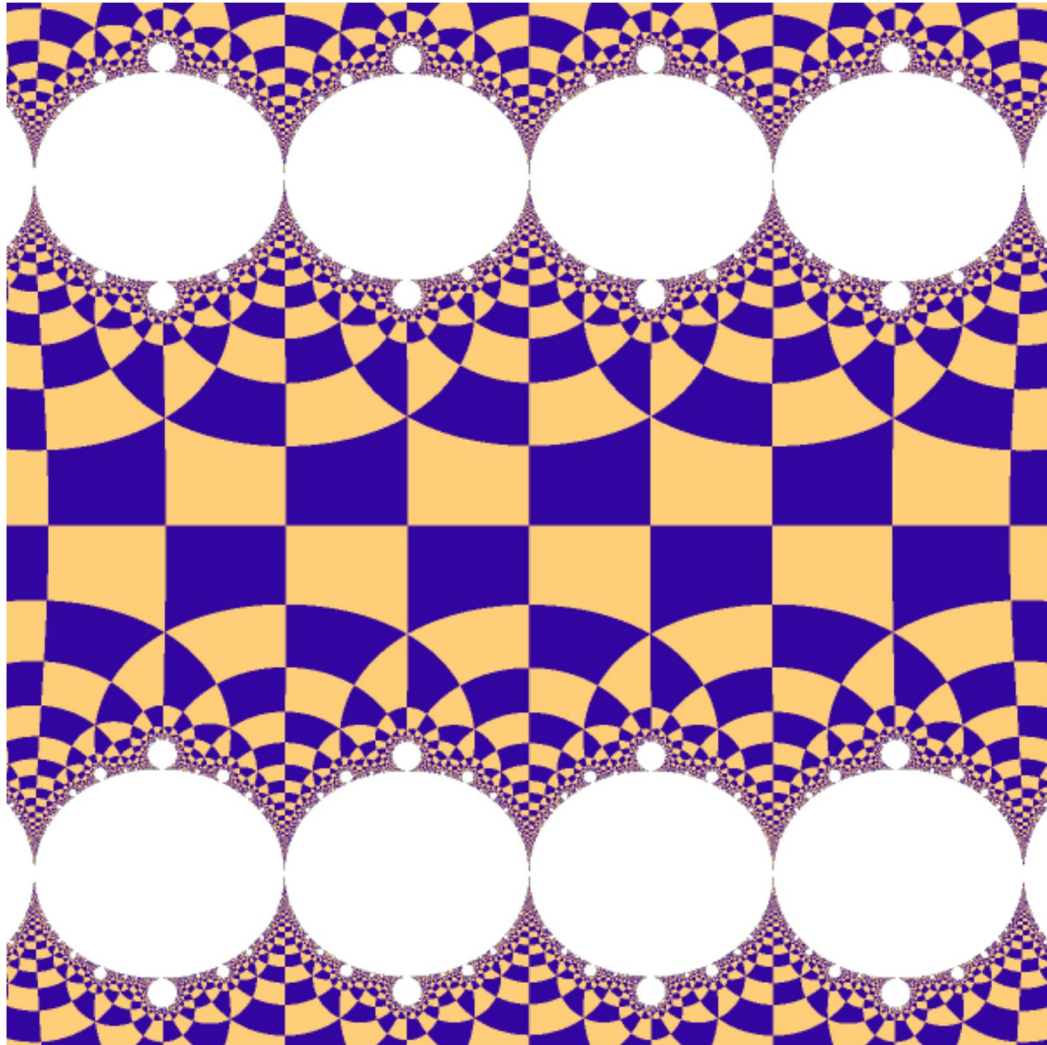
The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping



The mapping works equally well on arbitrary bended kaleidoscope.

Conformal mapping

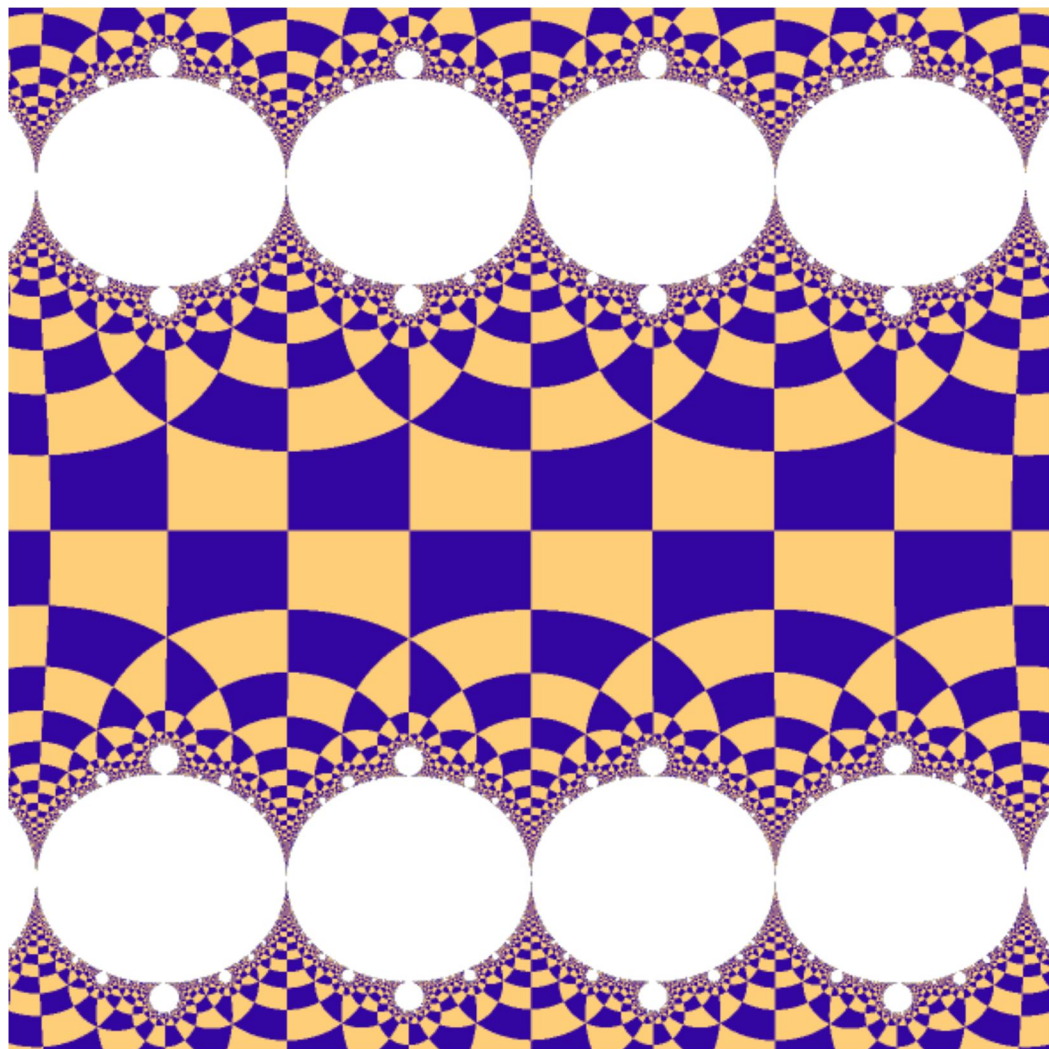


Final step.

We have pattern periodic in x and y-directions.

Pattern can be seamlessly wrapped around cylinder of appropriate radius.

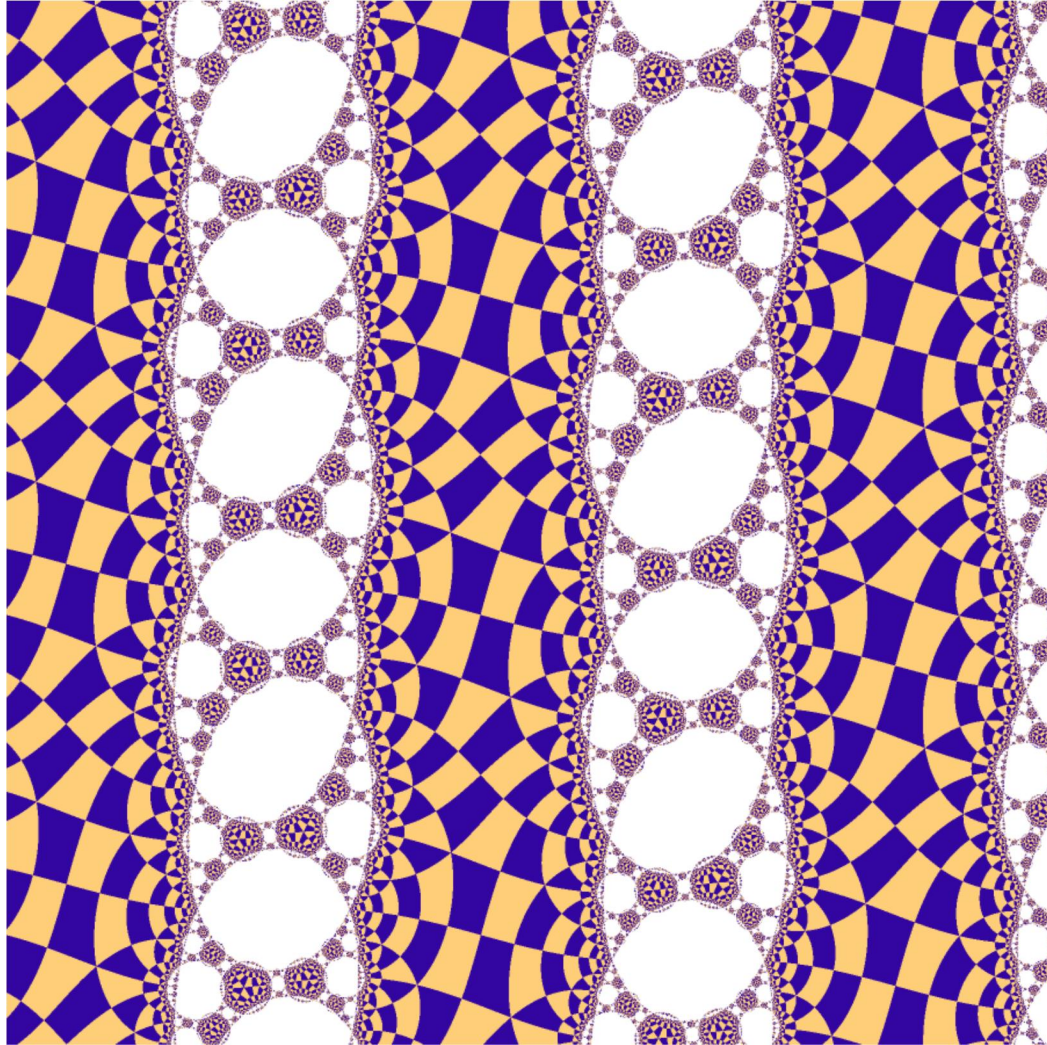
Conformal mapping



youtube animation

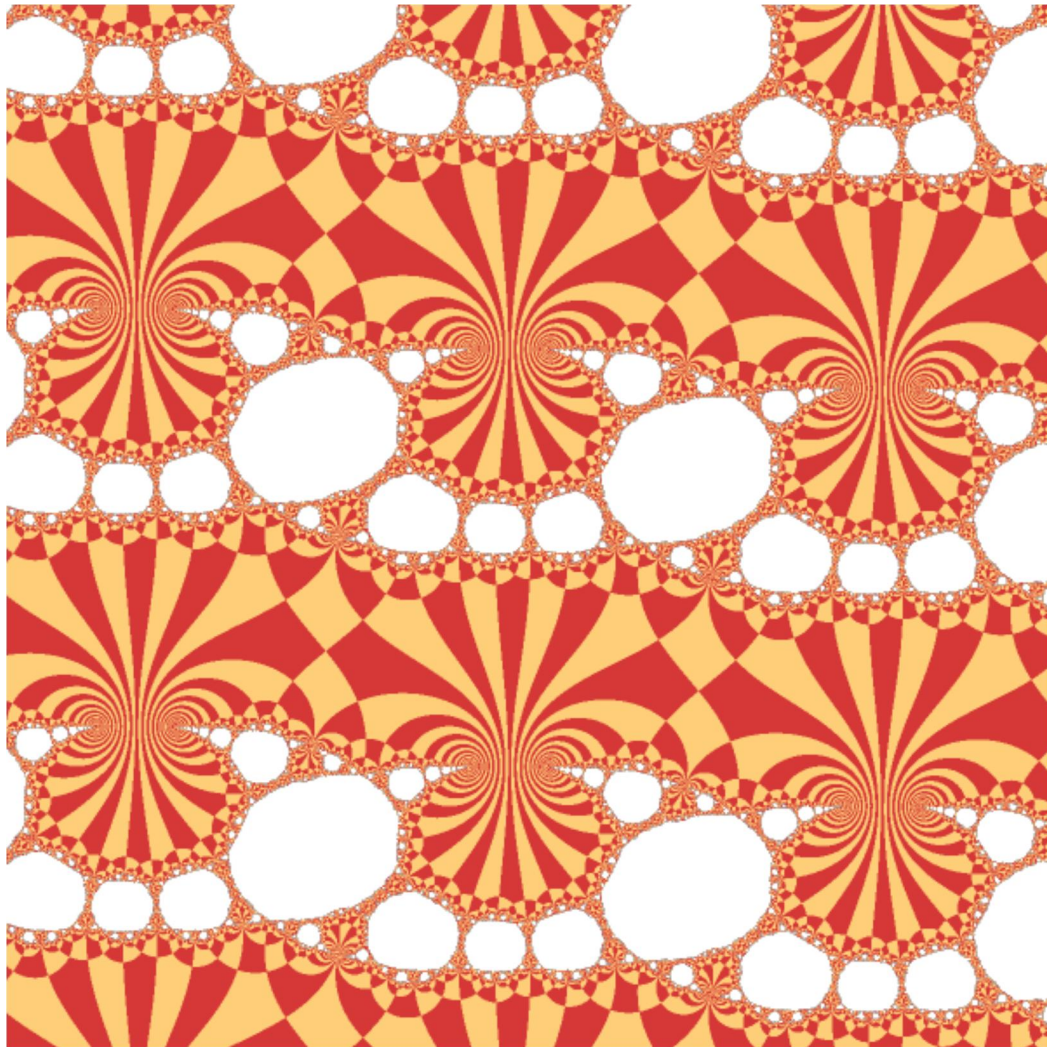
<http://youtu.be/ve6yYizwVQY>

Conformal mapping



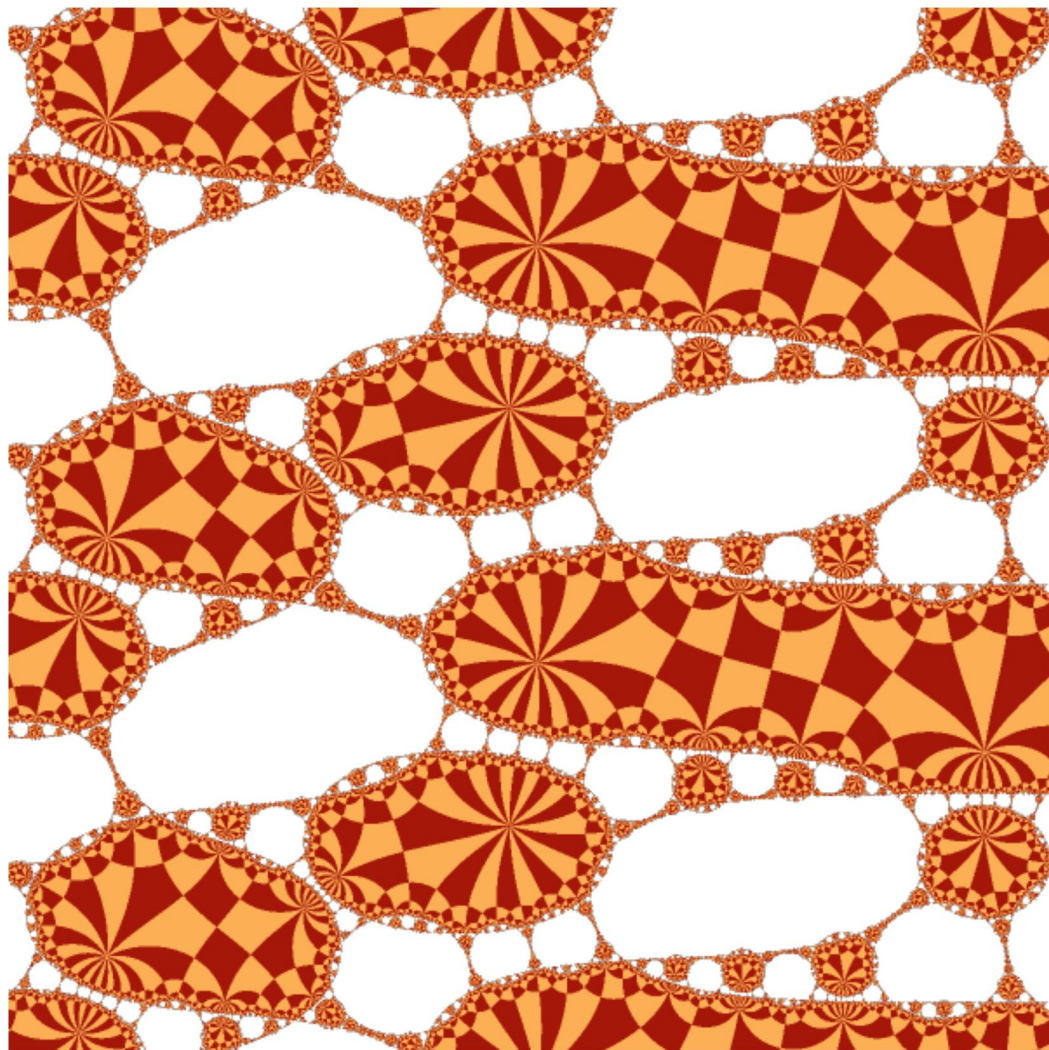
More wild conformally mapped
kaleidoscope.

Conformal mapping



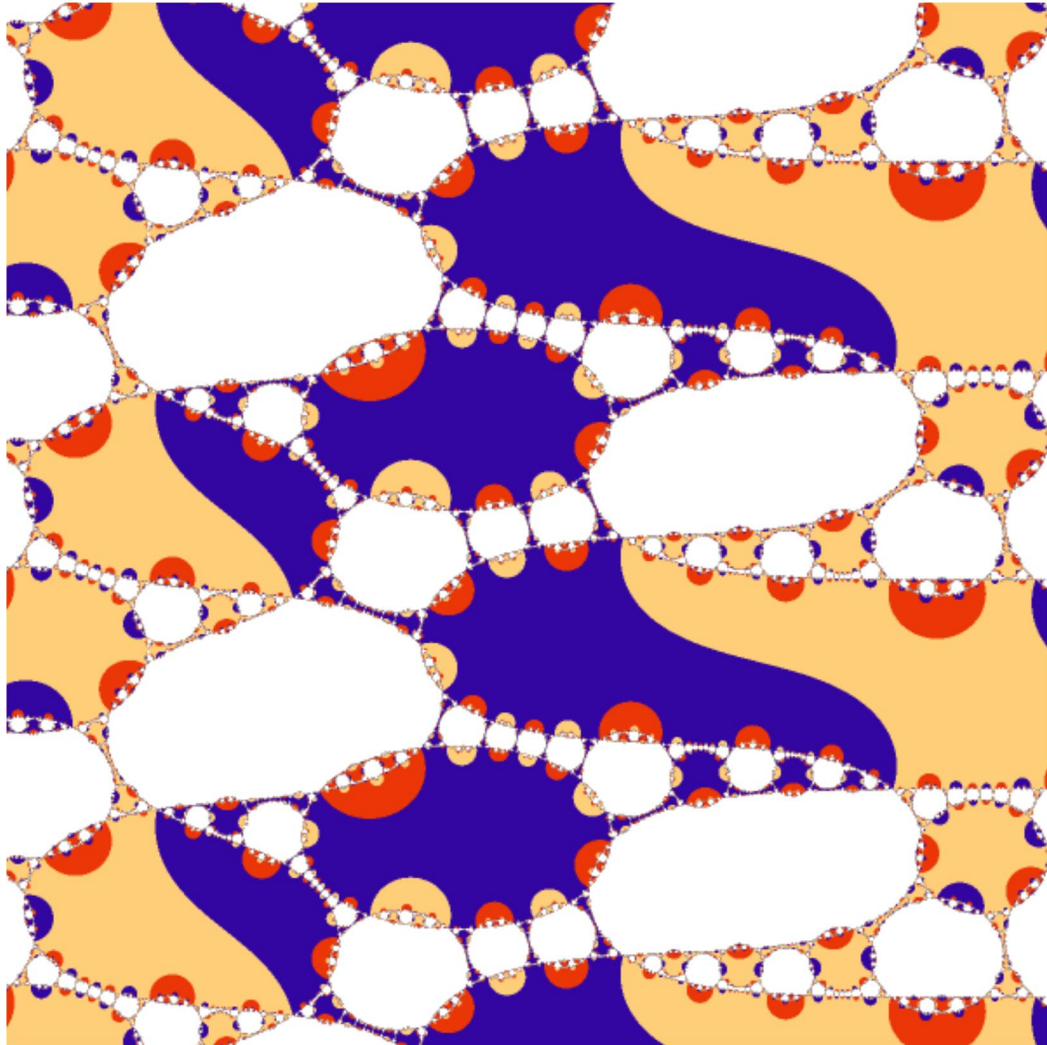
More wild conformally mapped
kaleidoscope.

Conformal mapping



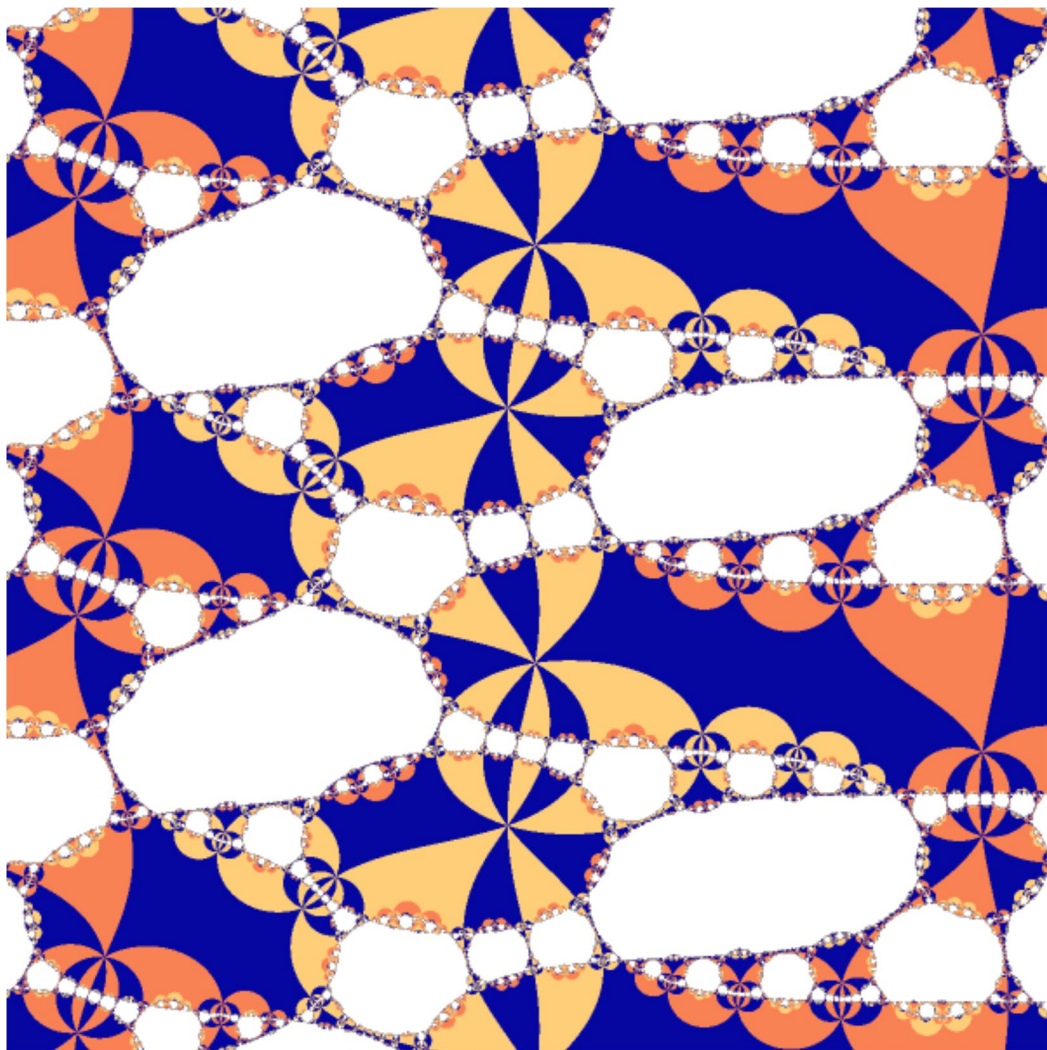
More wild conformally mapped
kaleidoscope.

Conformal mapping



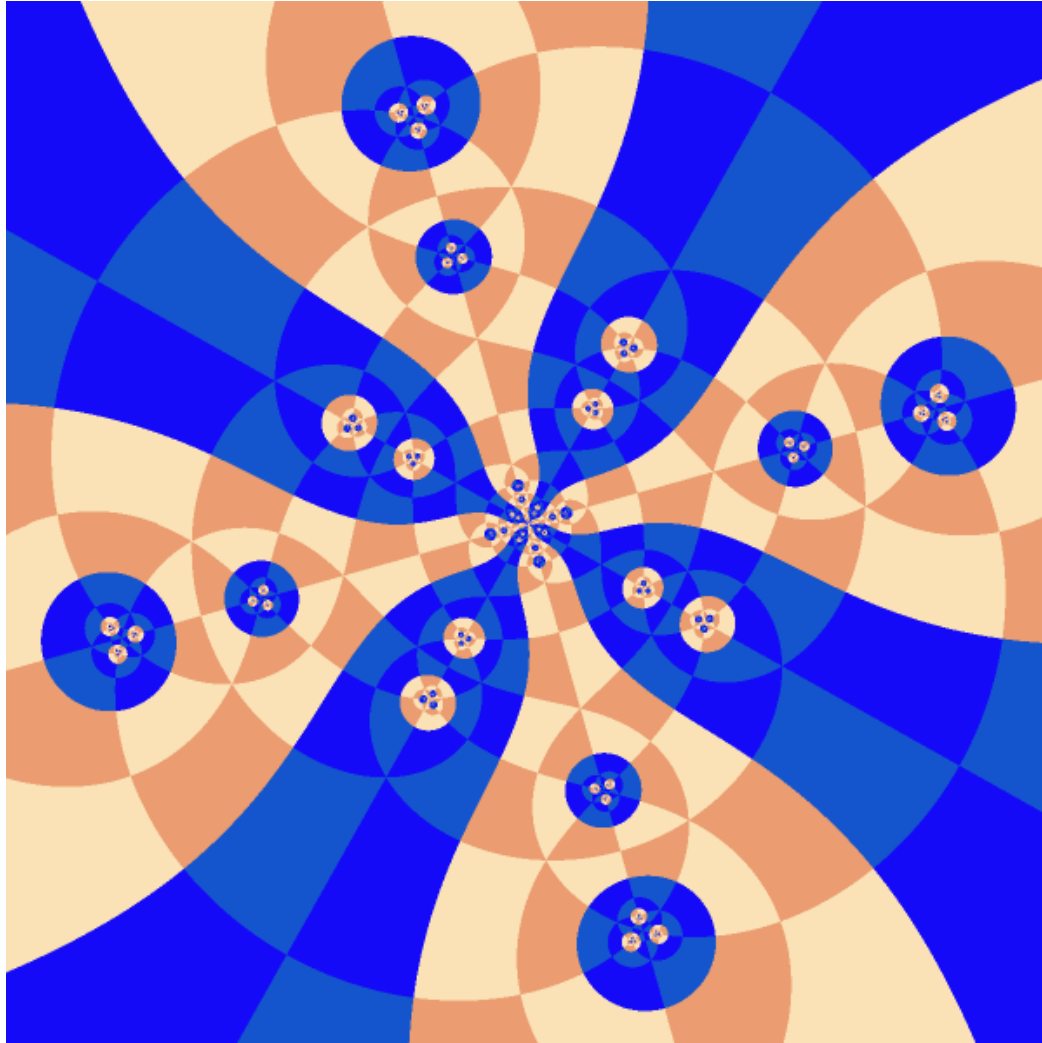
Different coloring of the same kaleidoscope

Conformal mapping



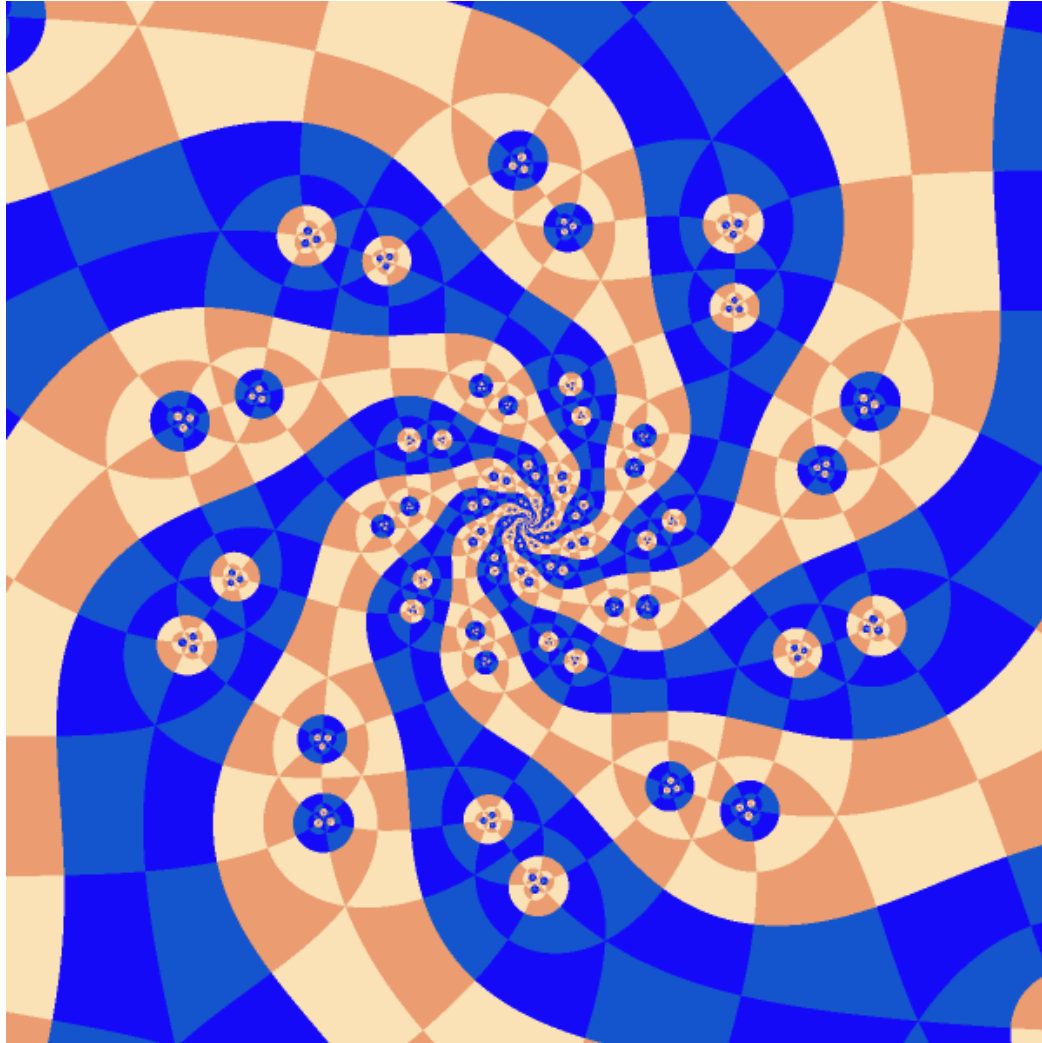
Different coloring of the same kaleidoscope

Conformal mapping



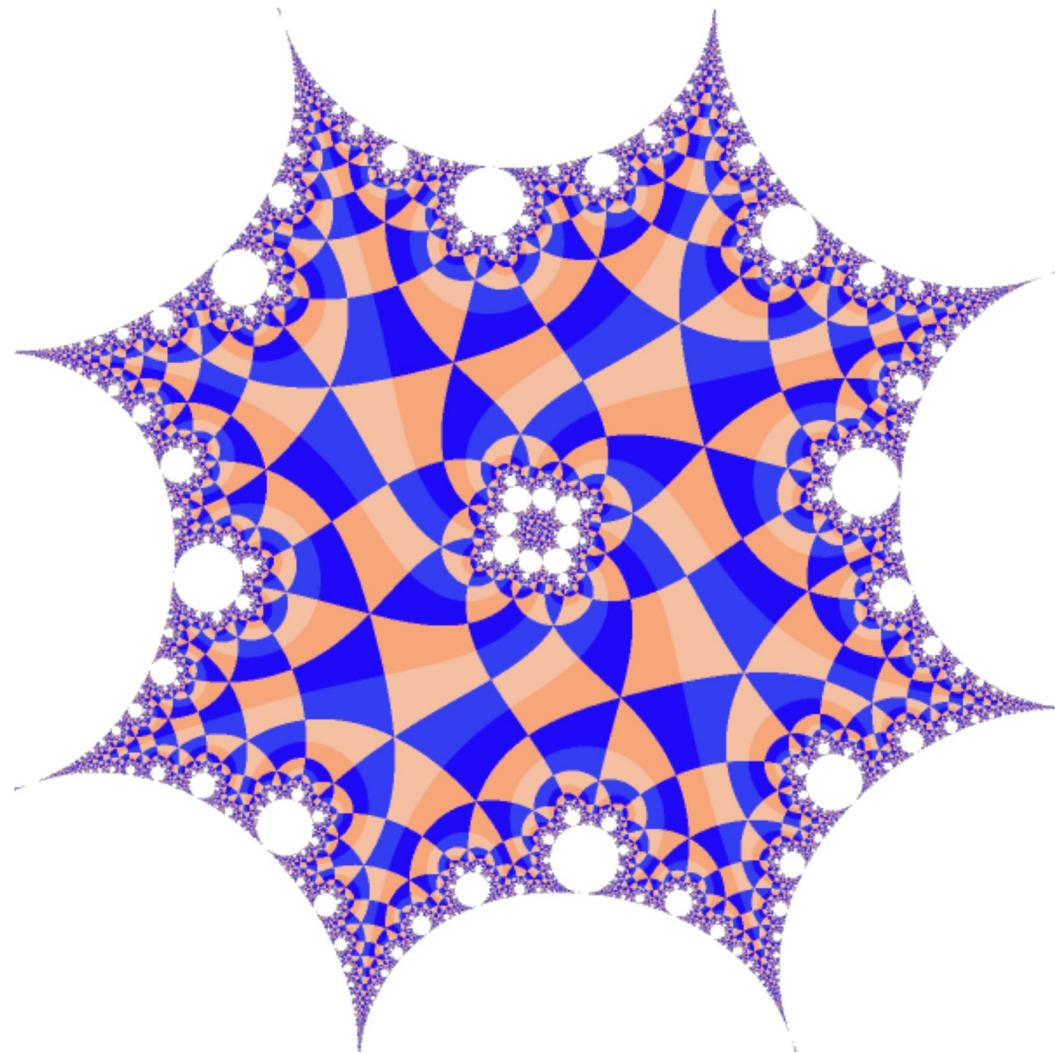
Band stretching combined with exponential mapping

Conformal mapping



Band stretching combined with exponential mapping

Conformal mapping



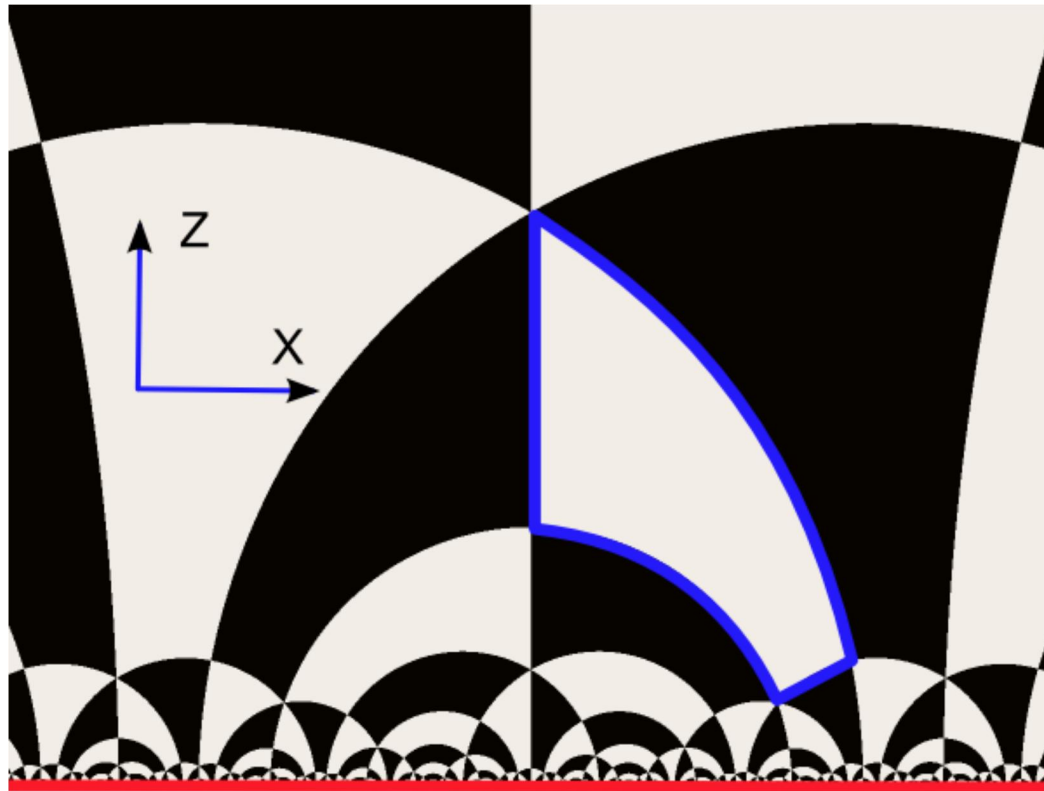
Band stretching combined with exponential mapping

Finite 3D tile size



What if we bend 4 sided chimney to extreme and make four sided tetrahedron of finite volume?

Finite 3D tile size



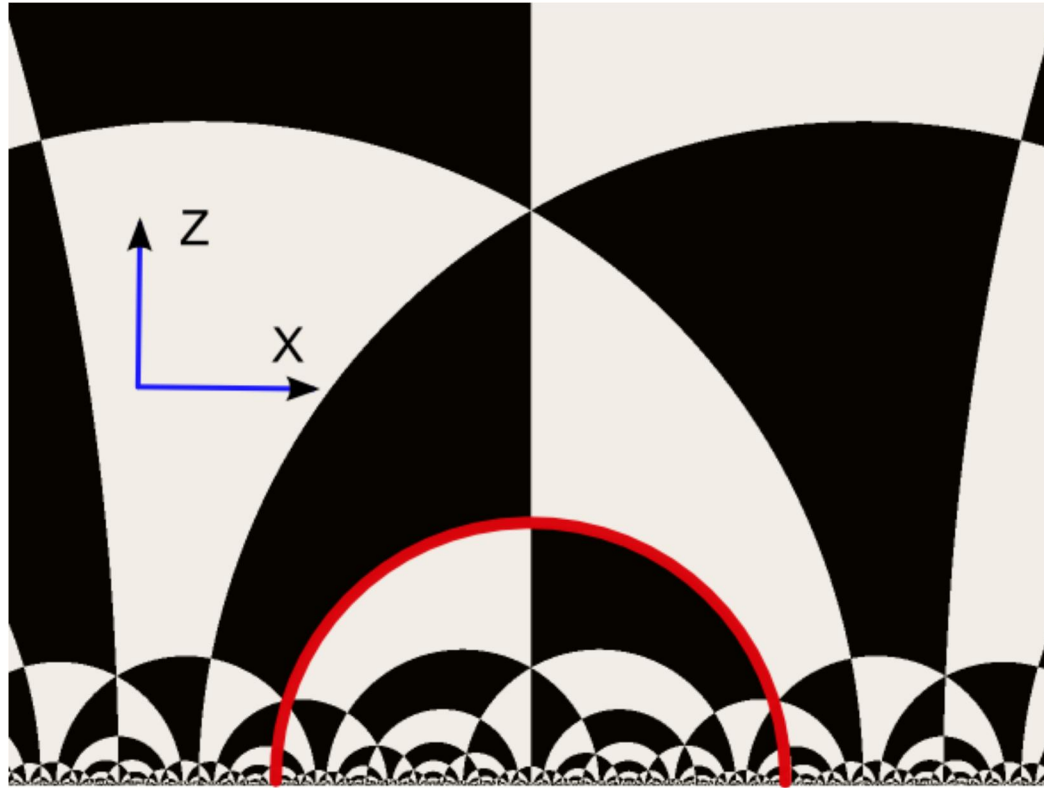
3D tiling with finite fundamental domain

This is XZ plane cross section of the tiling in the upper half space

Tile size tends to zero as we approach the upper half space boundary

Pattern on the boundary of the upper half space disappears

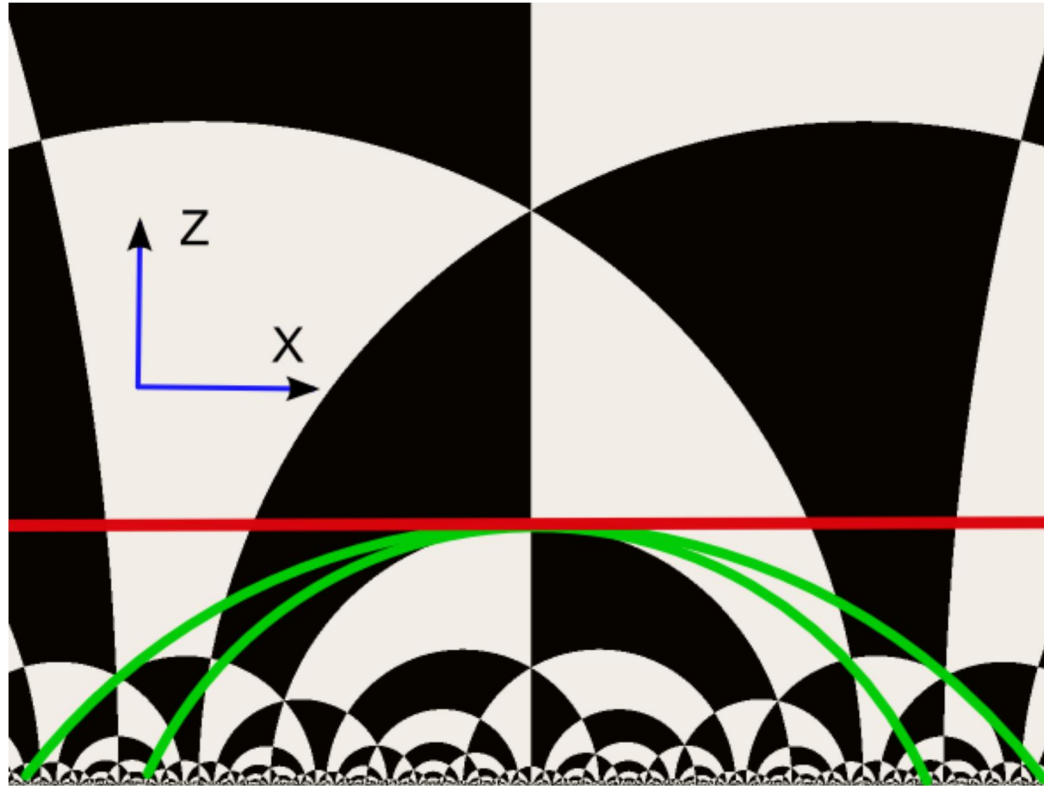
Finite 3D tile size



We can take cross section of the 3D tiling through some hyperbolic plane

This will give us something similar to an image of a 2D hyperbolic tiling in the Poincare disk model

Finite 3D tile size



We also can take cross section through series of hypercycles (green) passing via given point. The limit of such hypercycles is horosphere - spherical surface tangent to the boundary of the hyperbolic space.

Here the horosphere (red) touches the boundary at the infinity and it looks like euclidean plane orthogonal to z-axis.

Horosphere cross section

We start with cross section through hyperbolic plane. The tiny image of this cross section matched in scale with next series of cross sections through hypercycles and horosphere.



Horosphere cross section

Hypercycle cross section step 1



Horosphere cross section

Hypercycle cross section step 2



Horosphere cross section

Hypercycle cross section step 3

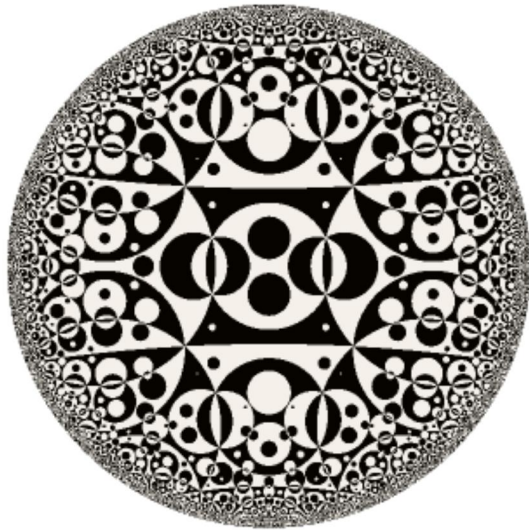


Horosphere cross section

Hypercycle cross section step 4

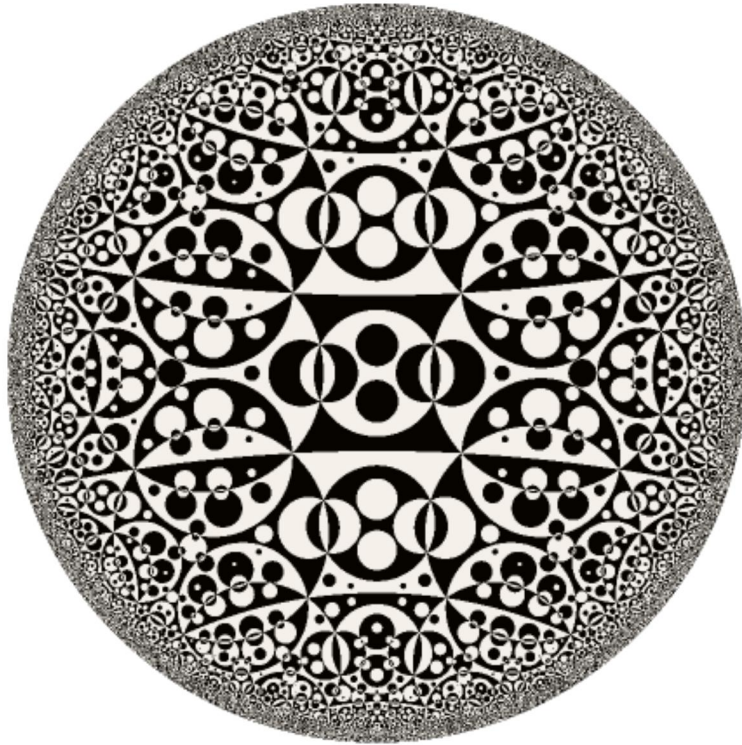


Horosphere cross section



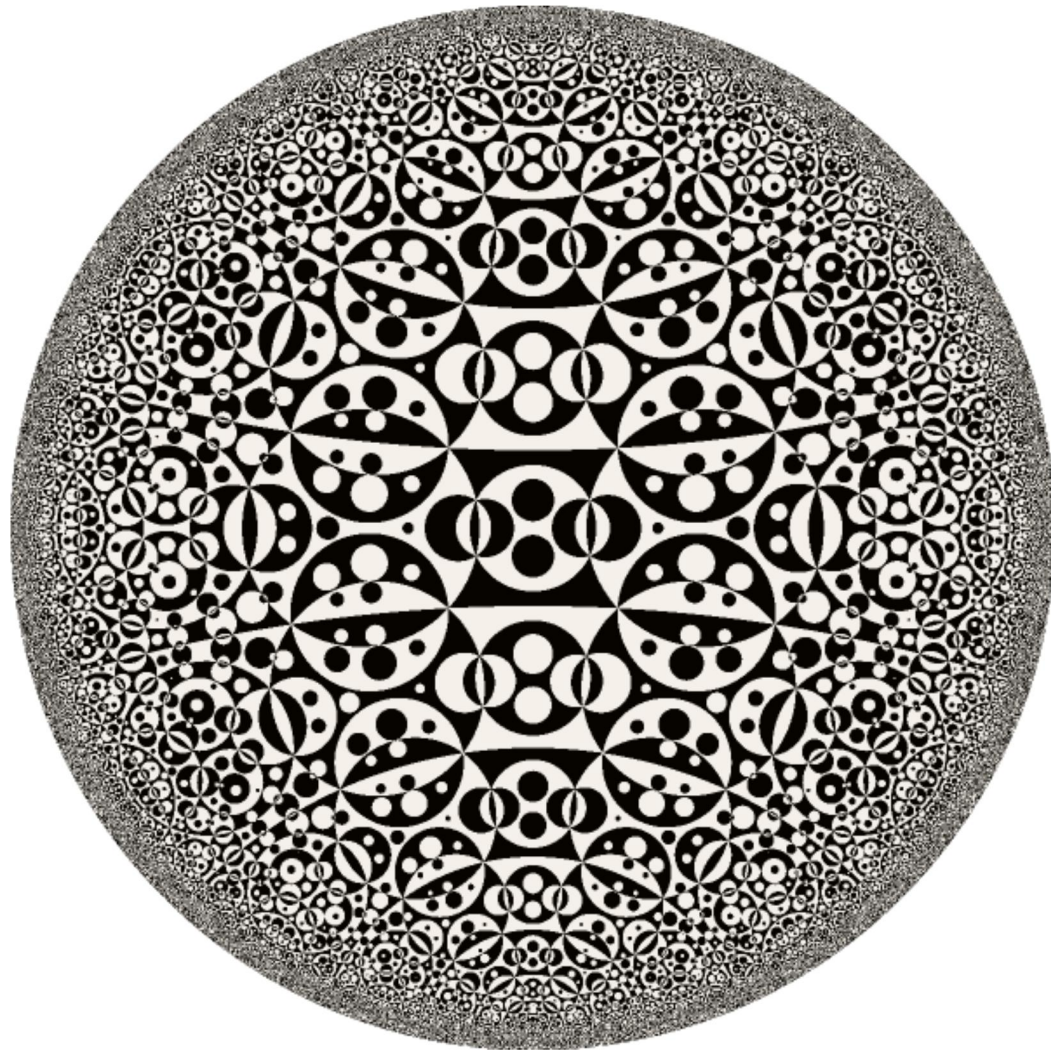
Hypercycle cross section step 5

Horosphere cross section



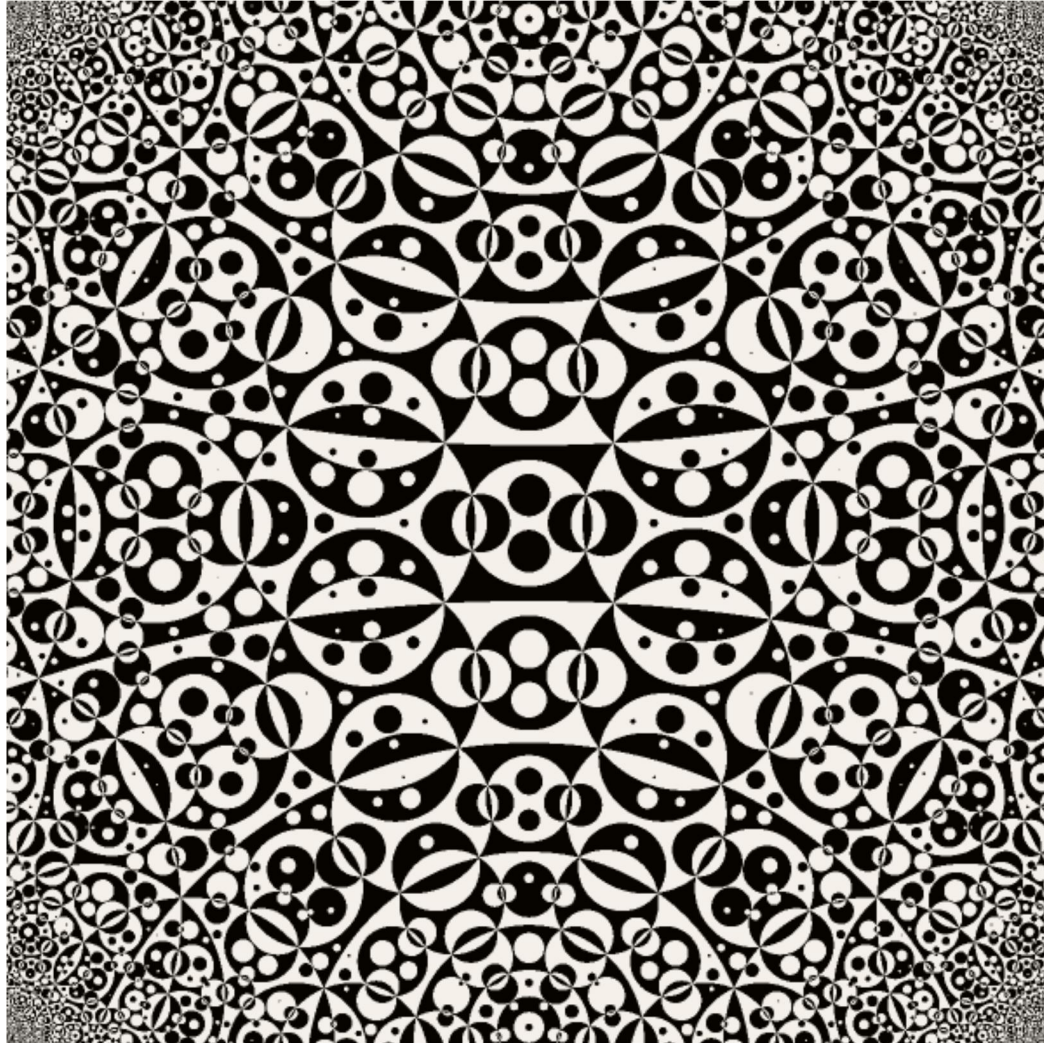
Hypercycle cross section step 6

Horosphere cross section



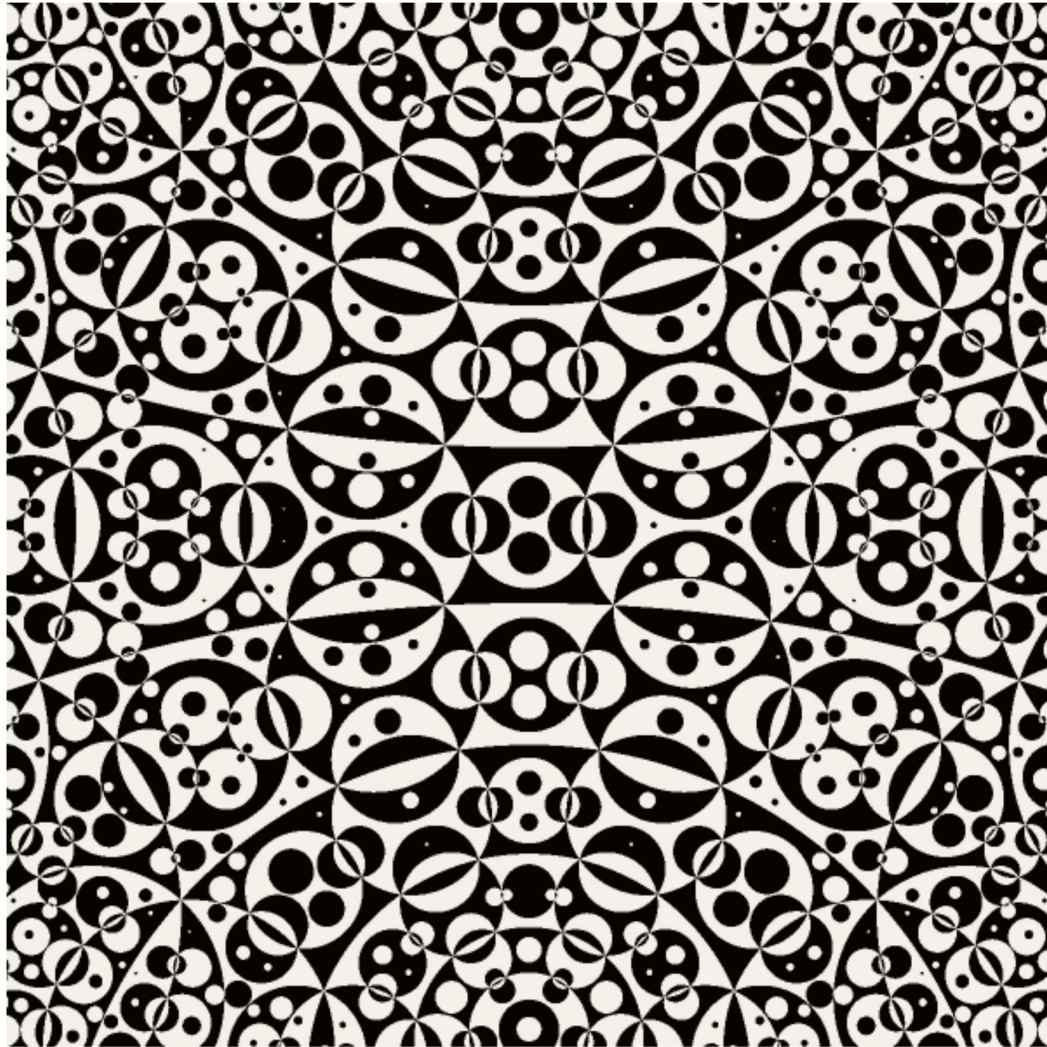
Hypercycle cross section step 7

Horosphere cross section



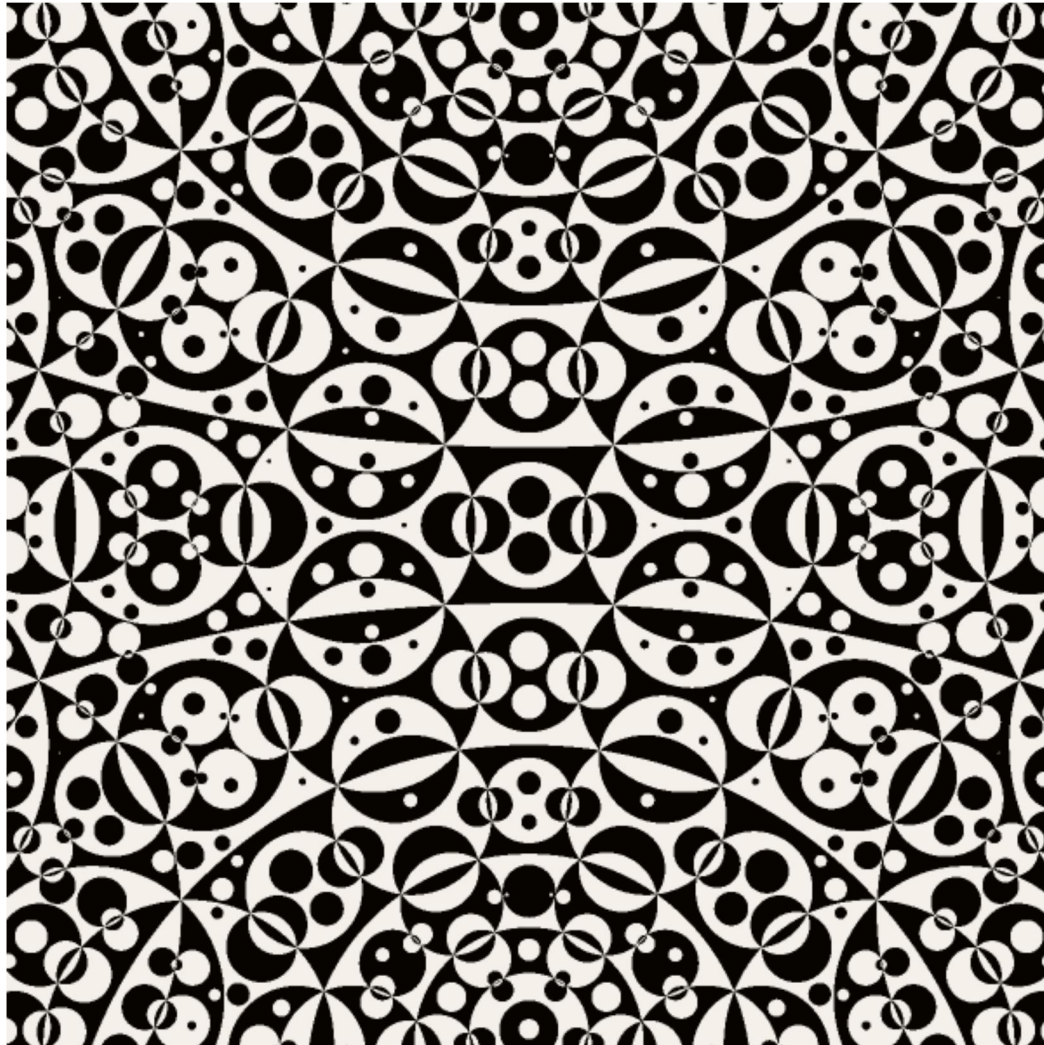
Hypercycle cross section step 8

Horosphere cross section



Hypercycle cross section step 9

Horosphere cross section

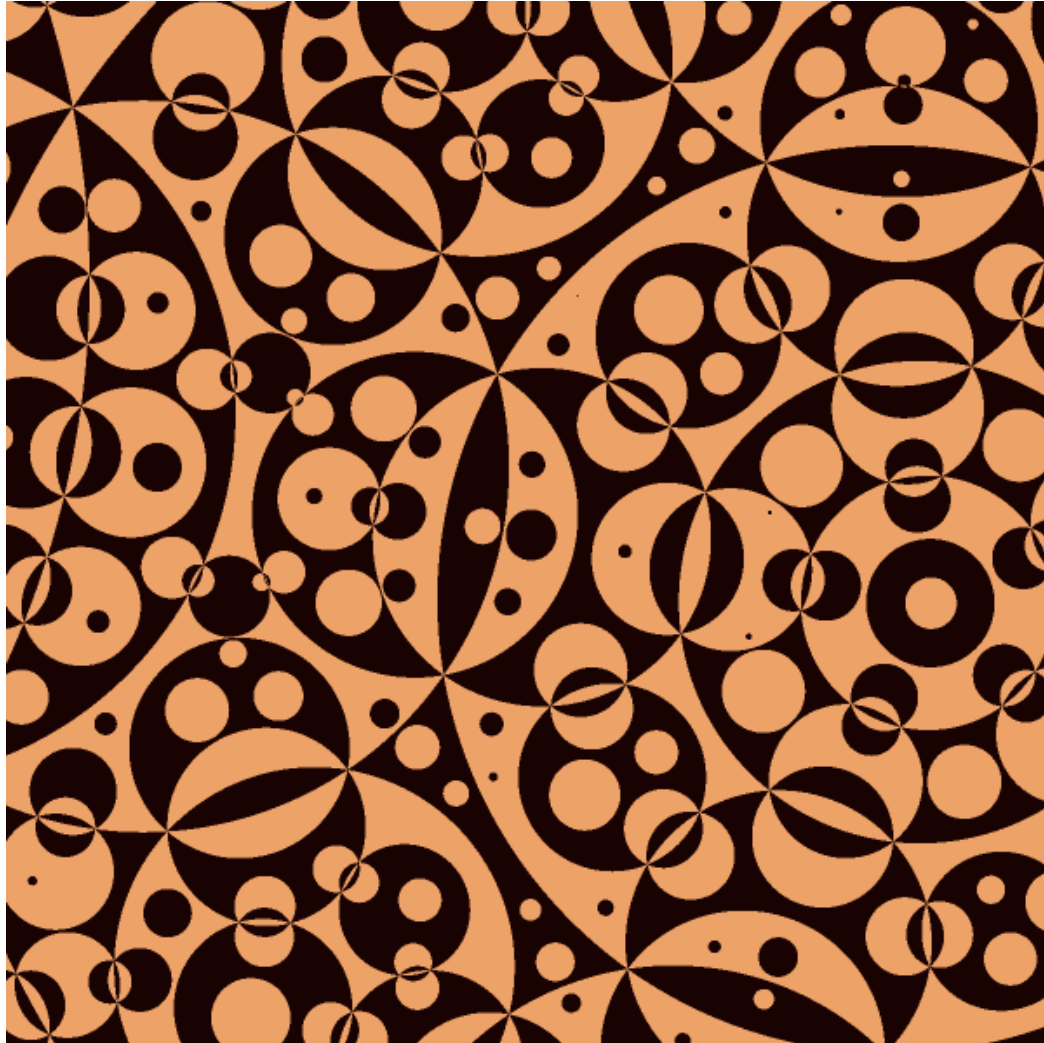


Hypercycle becomes horosphere. All the circles in the pattern are intersections of hyperplanes of the tiling with horosphere.

Hyperbolic metrics inside of horosphere becomes euclidean. Horosphere is in fact a model of the 2D euclidean geometry embedded in 3D hyperbolic space.

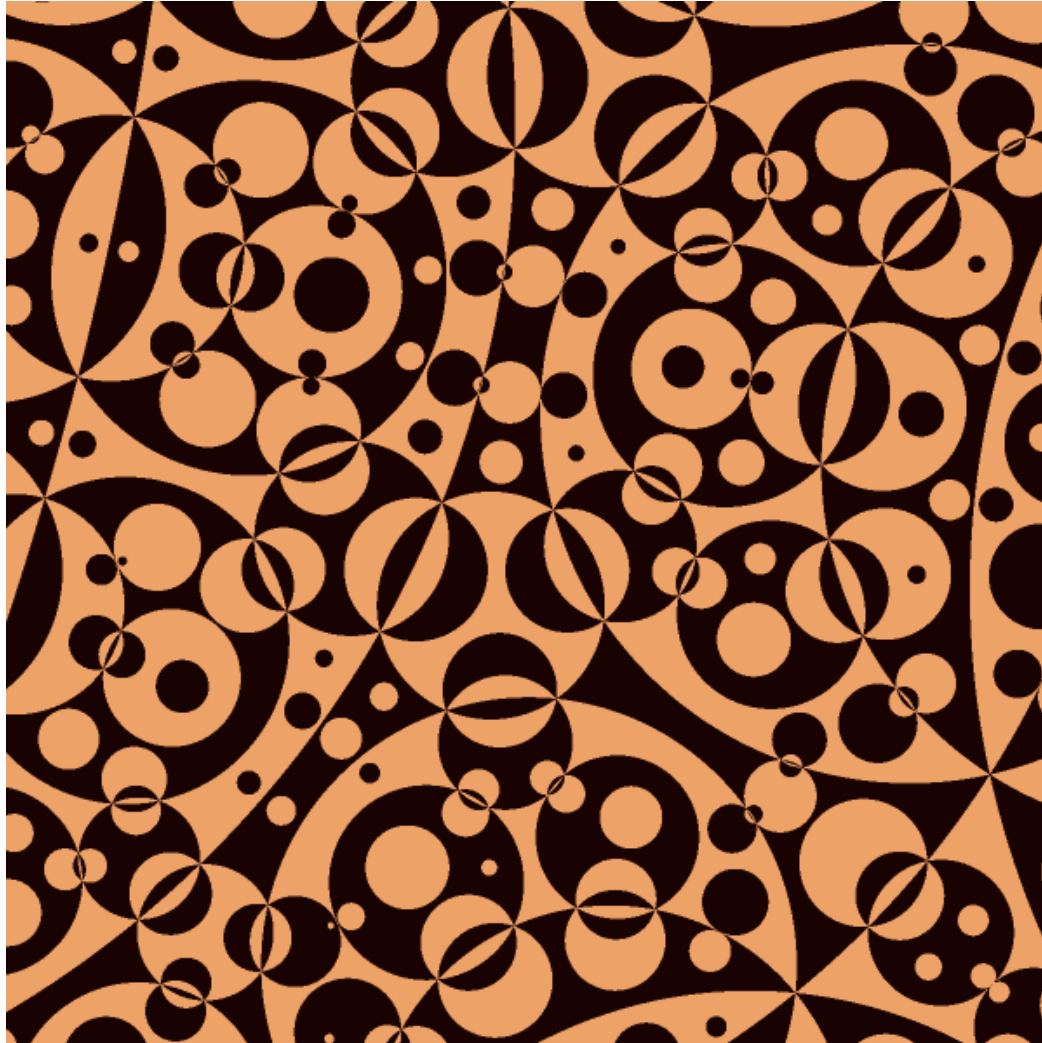
In contrast to the hyperplane or hypercycle cross sections the pattern in the horosphere cross section has constant scale. The pattern has no periods or repetitions. At the same time pattern looks similar everywhere. This effect is somewhat analogous to some non periodic tilings.

Horosphere cross section



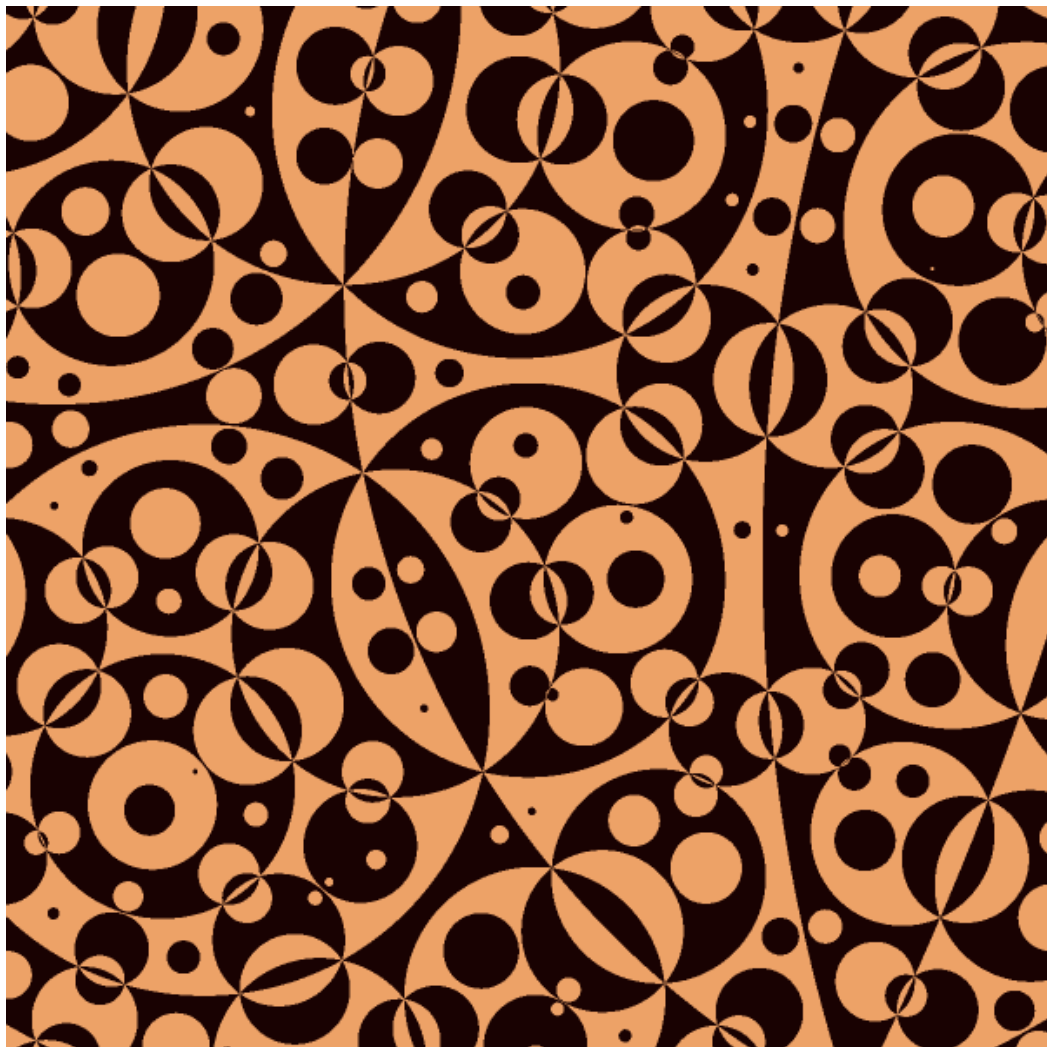
Pattern looks similar everywhere and is different everywhere.

Horosphere cross section



Pattern looks similar everywhere and is different everywhere.

Horosphere cross section



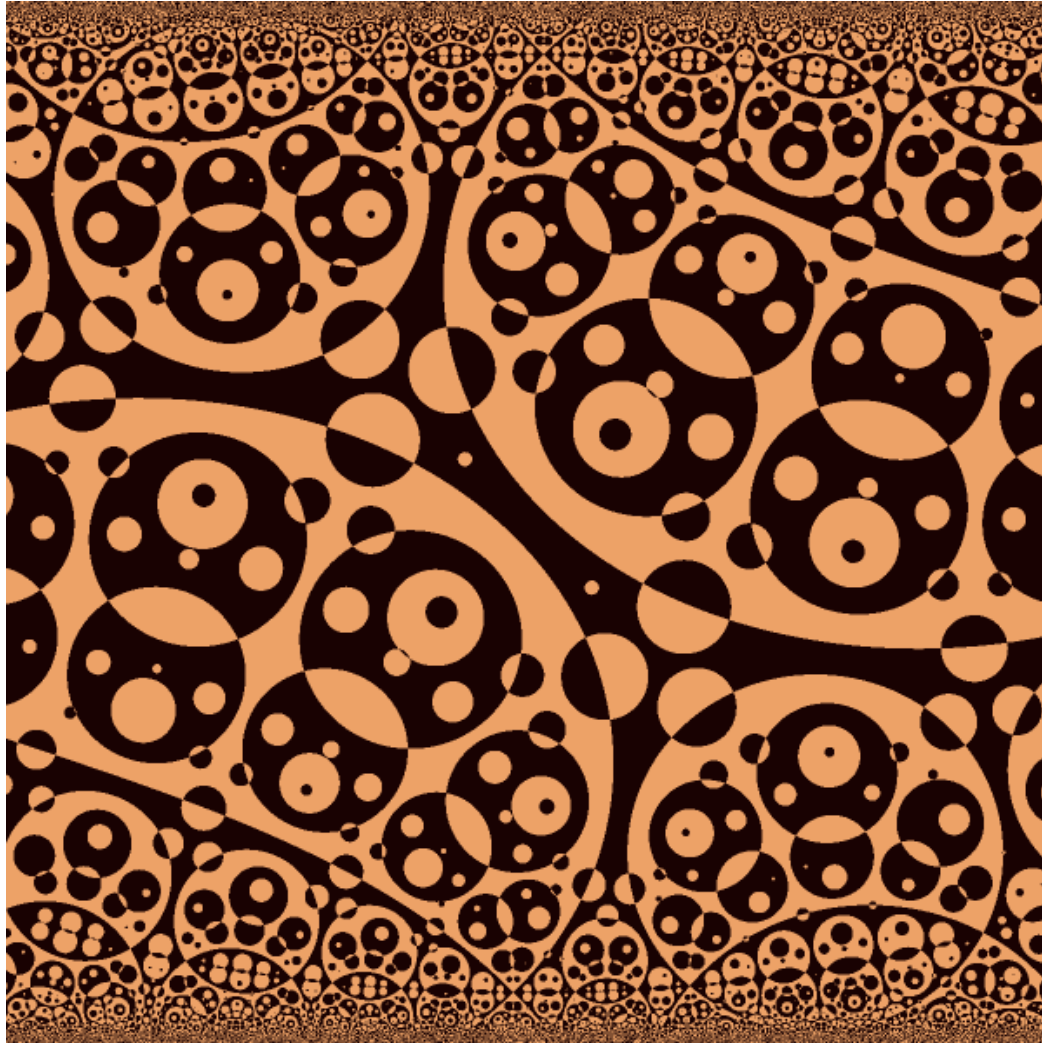
Pattern looks similar everywhere and is different everywhere.

Horosphere cross section



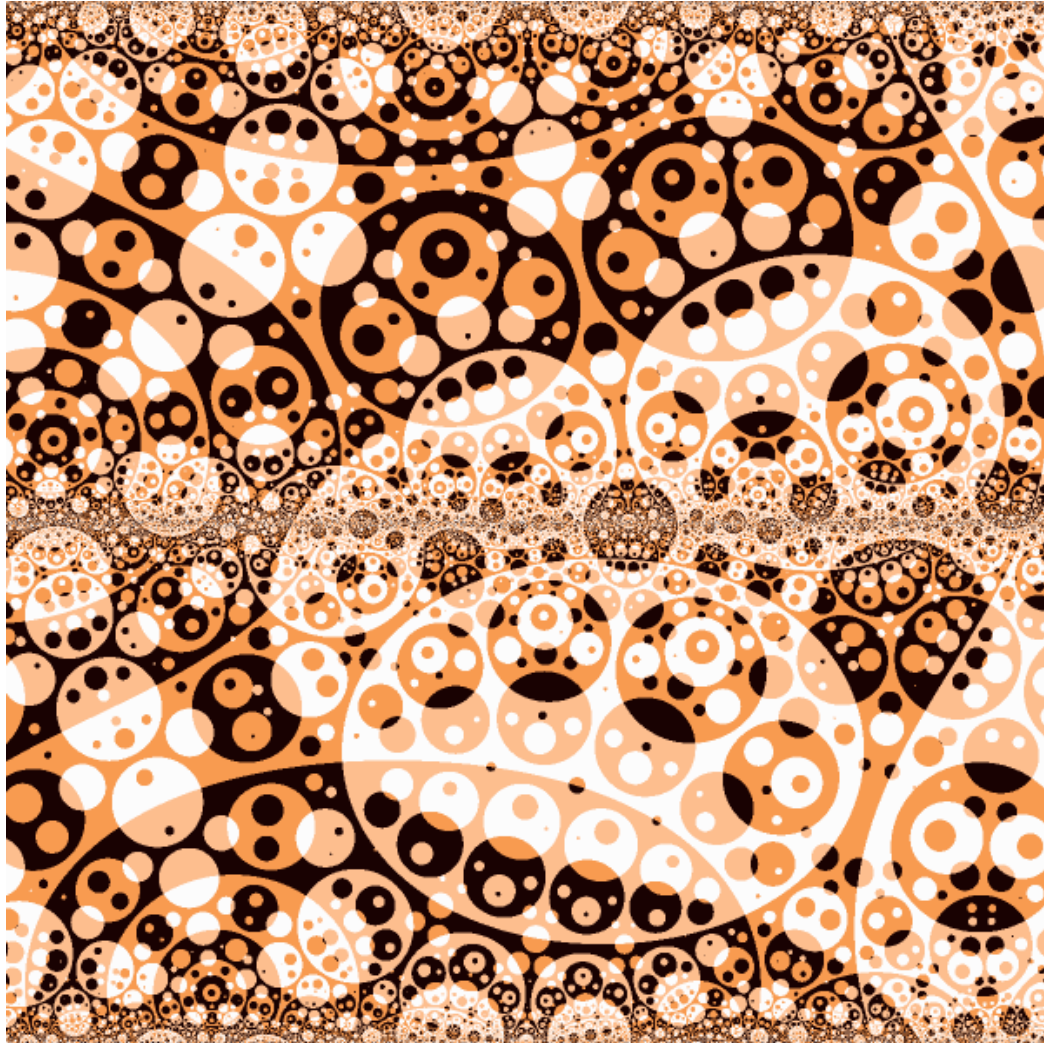
Pattern looks similar everywhere and is different everywhere.

Horosphere cross section



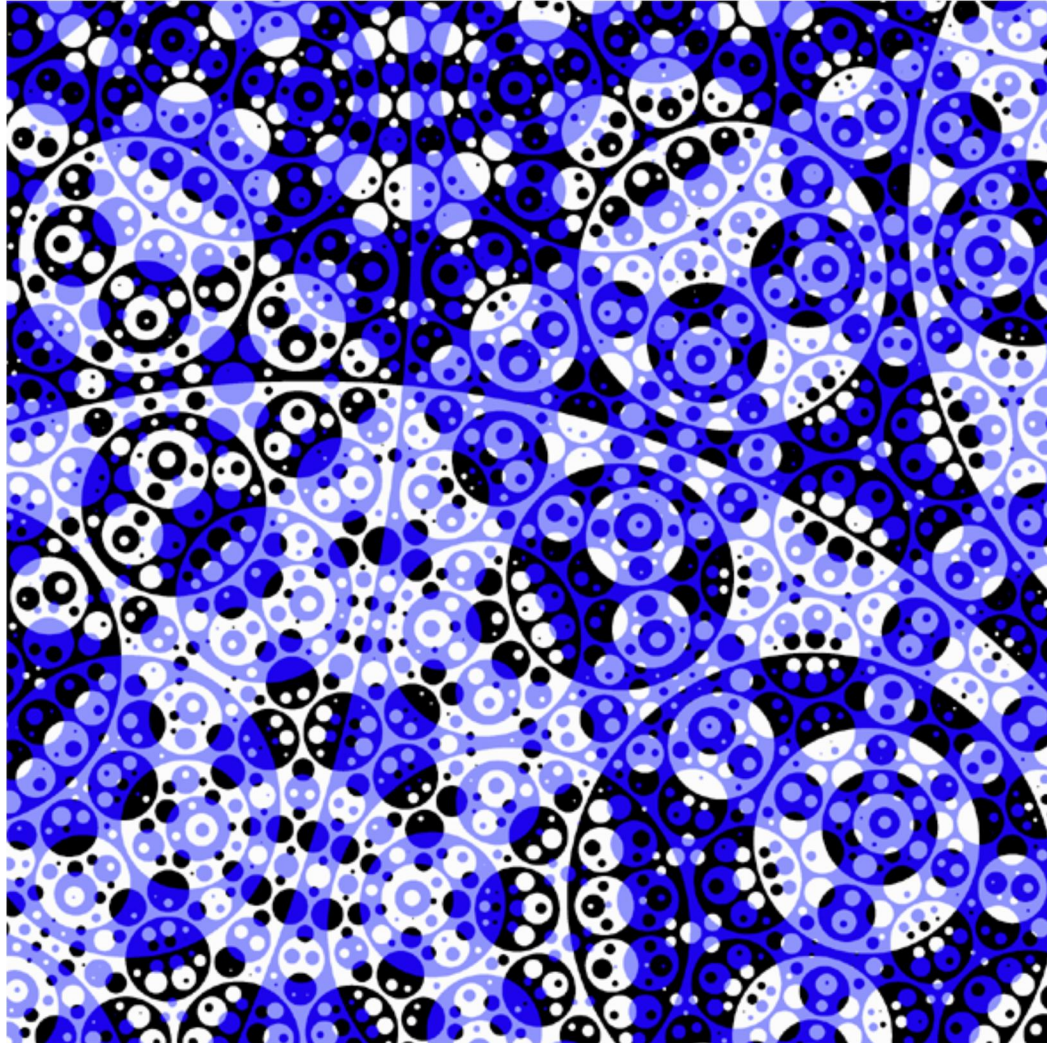
Combination of horosphere cross section
and conformal mapping

Horosphere cross section



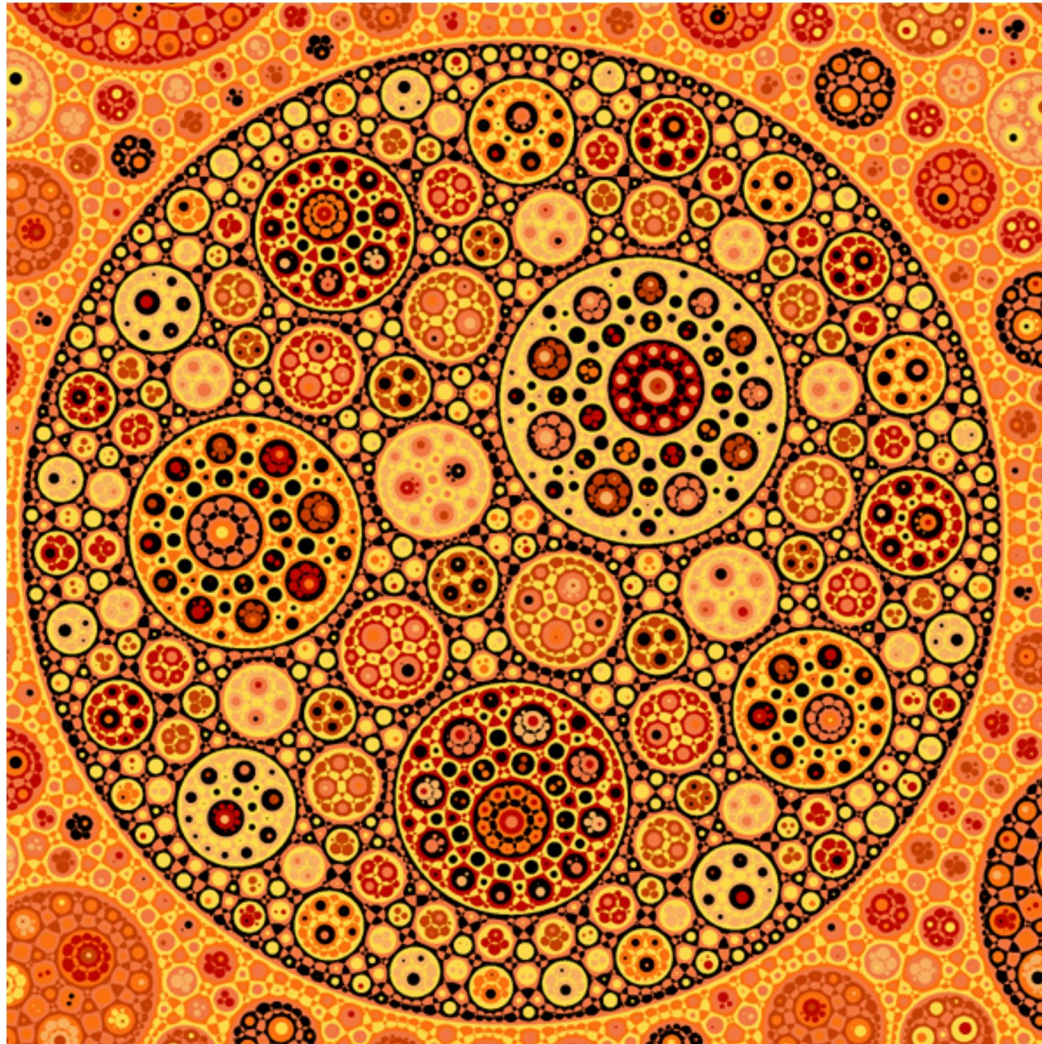
This pattern can be seamlessly wrapped around cylinder

Horosphere cross section



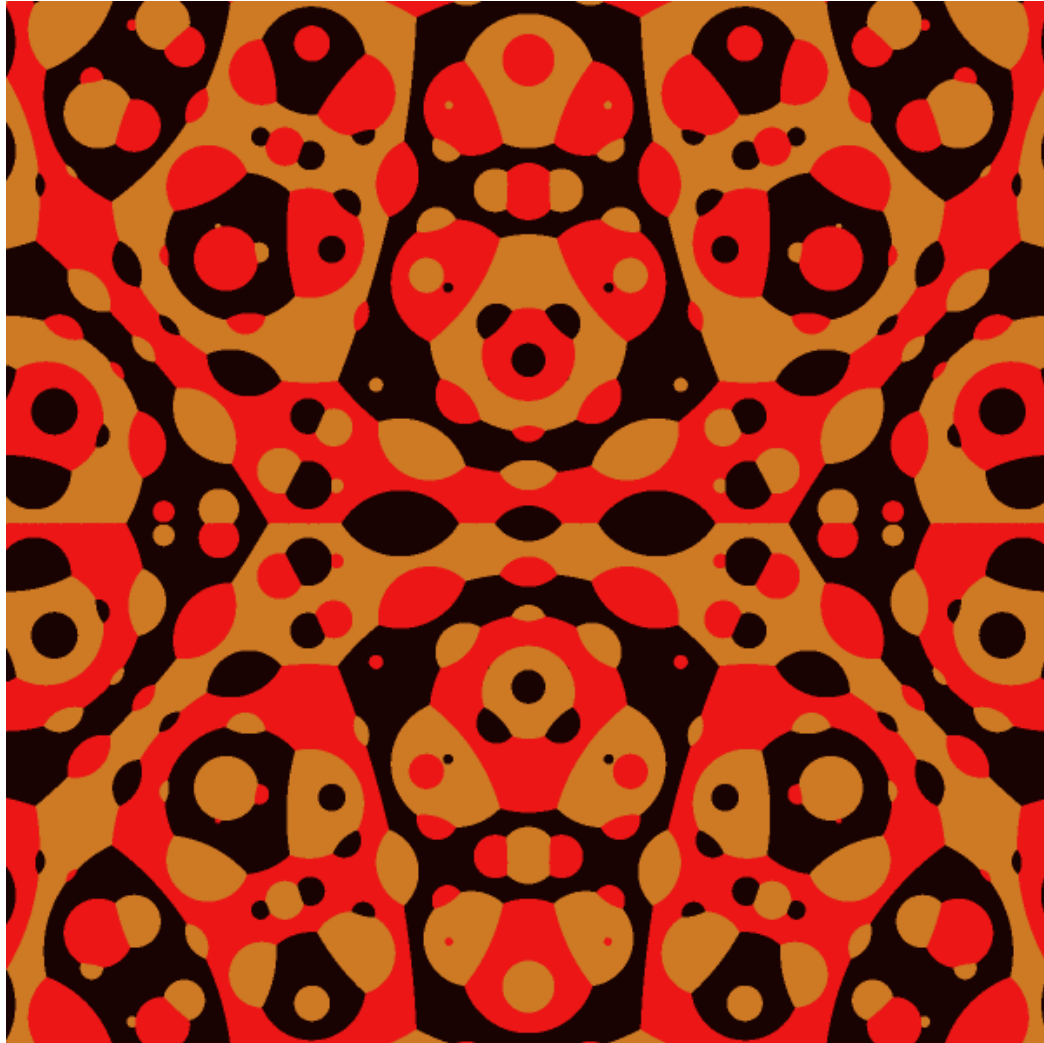
More horosphere cross section

Horosphere cross section



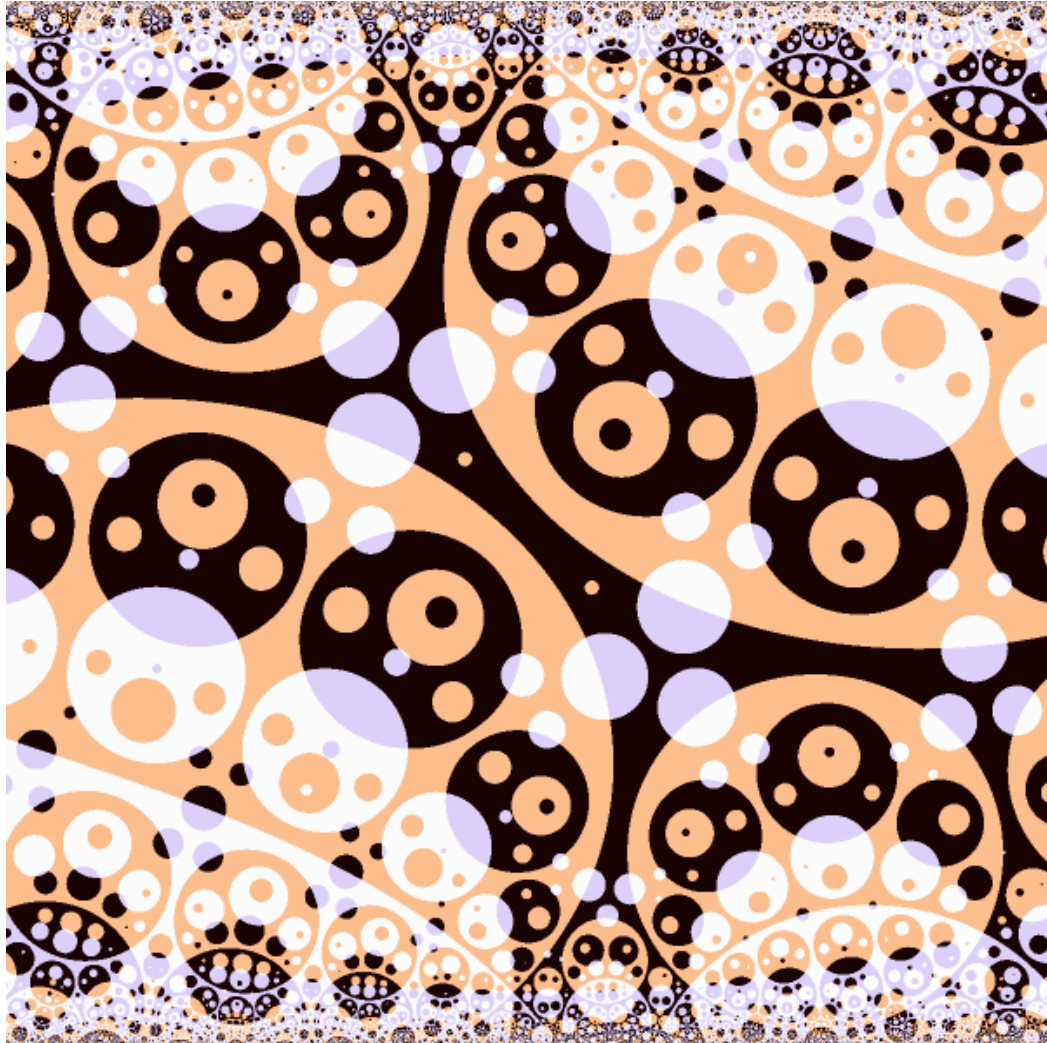
More horosphere cross section

Horosphere cross section



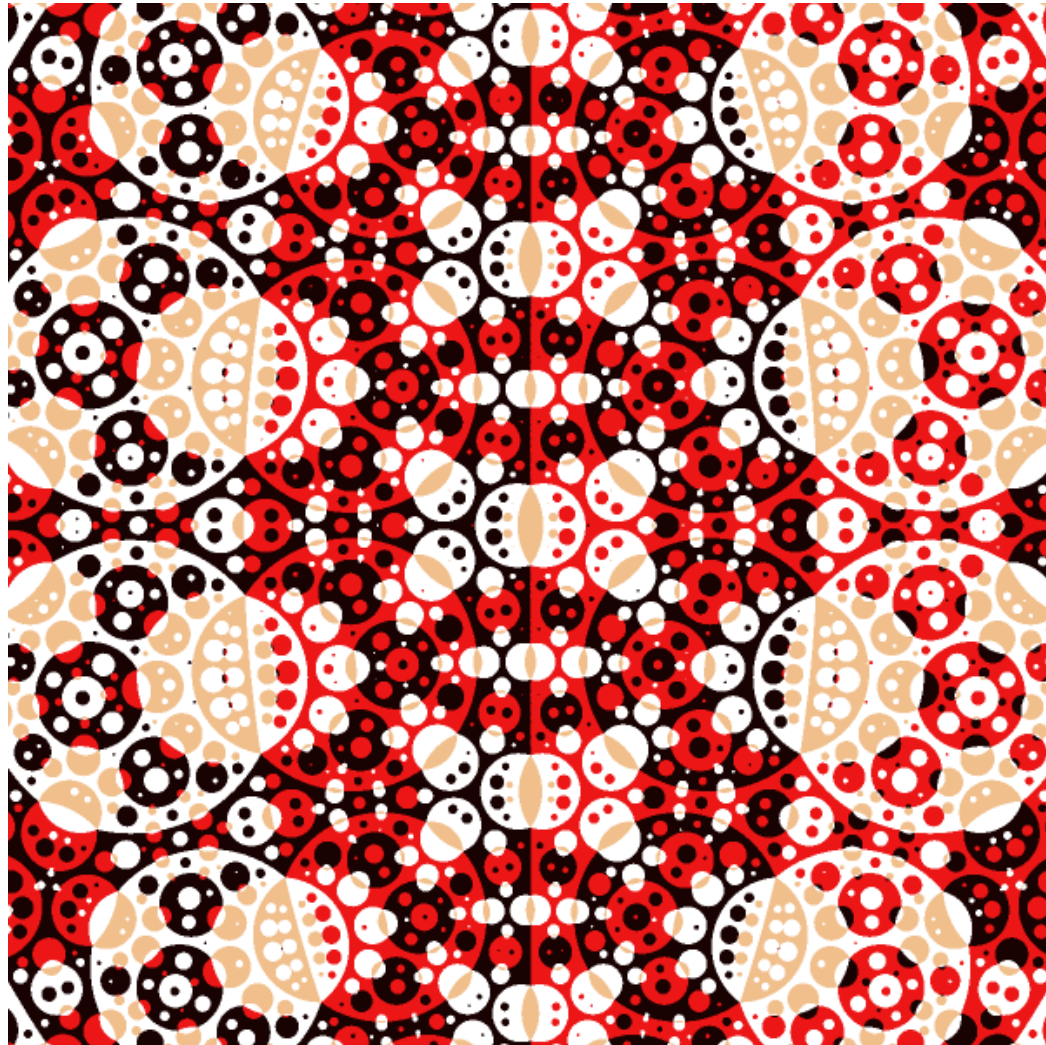
More horosphere cross section

Horosphere cross section



More horosphere cross section

Horosphere cross section



More horosphere cross section

Horosphere Animation 1



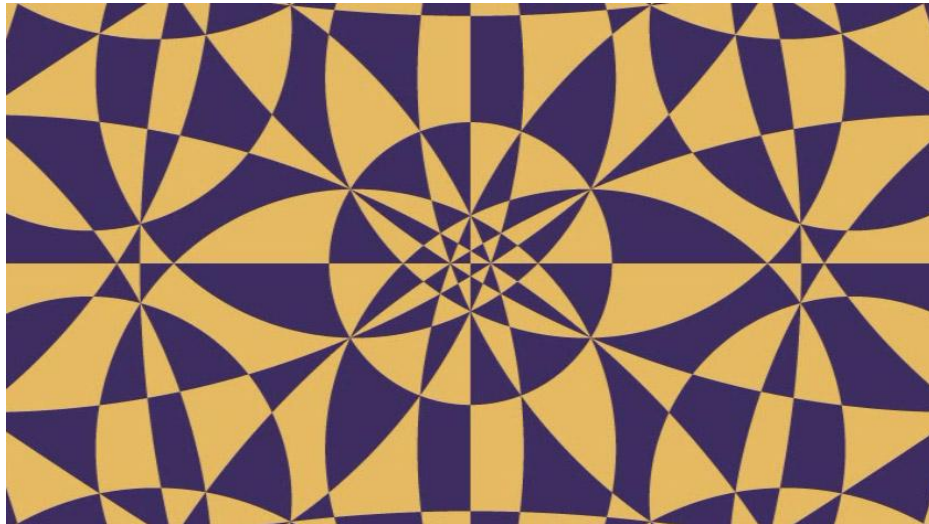
Fly through the hyperbolic tiling cutting the tiling with horosphere.

Different colors are used for different cosets of a subgroup of the symmetry group.

youtube animation

<http://www.youtu.be/KrAqh48-imA>

Horosphere Animation 2



Fly through the hyperbolic tiling cutting the tiling with horosphere.

Different colors are used for different cosets of a subgroup of the symmetry group.

youtube animation <http://www.youtu.be/35ALc2xUR4Q>

Horosphere Animation 3



Fly through the hyperbolic tiling cutting the tiling with horosphere.

Different colors are used for different cosets of a subgroup of the symmetry group.

youtube animation

<http://www.youtu.be/FDwhrFDIaOI>

End



Online address of the talk:
bulatov.org/math/1107/